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NAPIER
TERCENTENARY MEMORIAL VOLUME

EDITED BY

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PREFACE

The Addresses and Essays contained in this volume were communicated to the International Congress which met in Edinburgh towards the end of July 1914, to commemorate the Tercentenary of the publication of John Napier's *Mirifici Logarithmorum Canonis Descriptio.*

These communications naturally fall into two groups, according as they treat of the life and work of Napier himself, or deal with subsequent developments of the logarithmic idea which will ever be associated with the name of the Baron of Merchiston.

From the historic point of view the former group will be found particularly interesting. The various contributions have been prepared absolutely independently, each author following out his own line of thought without any knowledge of the intentions of the other contributors. Inspired though these essays are by consideration of the life and work of the same great man, and based largely upon the same limited set of facts, they show a surprising diversity both in point of view and mode of treatment. They are a striking tribute to the fertility of the idea first promulgated by Napier. At the same time they focus attention anew upon the authoritative statements in the writings of Napier and his contemporaries, and dispose once and for all of certain misapprehensions which have found currency in mathematical literature.

The remaining communications deal with a great variety of topics, the one common feature being calculation.

The illustrations are largely reproductions in facsimile of the title-pages and other portions of Napier's published books. An added interest is the presentation of similar parts of Bürgi's *Progress Tabullen*, one of the few extant copies of which was kindly lent to the Tercentenary Exhibition by the Town Library of Danzig.

The frontispiece is a reproduction in colour of the life-size portrait of John Napier in the possession of the University of Edinburgh; we desire to thank the University Court for permission to make the reproduction.

Our thanks are also due to Miss Napier of Chelsea for her permission to make a reduced copy in colour of the landscape of Merchiston Castle, as it appeared in the eighteenth century, and probably very much as it was in John Napier's own day.

The Head and Tail Pieces occasionally used throughout the volume are from the original edition of the *Descriptio*, published by Andrew Hart. The blocks were kindly supplied by Mr W. Rae Macdonald.

For the Napier Shield and heraldic design on the Cover we are indebted to the advice of Sir J. Balfour Paul, Lyon King of Arms, and to the skill of the Artist, Mr Graham Johnston.

As regards the Congress itself it is pleasant to recall the goodwill and friendliness which characterised its meetings, attended though these were by men and women whose nationalities were fated to be in the grip of war before a week had passed.

CARGILL GILSTON KNOTT,
Secretary and Editor.

THE ROYAL SOCIETY
OF EDINBURGH, 1915.

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INAUGURAL ADDRESS

BY

LORD MOULTON

THE INVENTION OF LOGARITHMS ITS GENESIS AND GROWTH

I feel fully sensible of the honour that has been done me in appointing me to address you on this occasion, the celebration of the Tercentenary of the Invention of Logarithms by Baron Napier of Merchiston. That name is a glory to the land and its bearer stands prominent among that small band of thinkers who by their discoveries have substantially increased the powers of the human mind as a practical agent. On such an occasion as this, when the personality and work of Napier are attracting world-wide interest, it is a flattering, but I fear unwise, preference that has been shown to me, and it behoves me to respond to it to the best of my ability.

But the fact that, of late, public interest has been concentrated upon Napier does not lessen the difficulty of my task to-day but rather adds to it. During the last few months, in well-nigh every centre of learning, discourses have been given and papers read on his life and works. Every fact known about him has become a commonplace. You are all aware that he was born in 1550 and died in 1617, that he wrote a formidable treatise on the Apocalypse, that he was suspected of black magic, that he proposed wonderful and futile methods to defeat the King's enemies, etc. etc. Probably by virtue of my position at this moment, I should be entitled to inflict on you anew all these historical fragments. But I should feel ashamed to do so. I can tell you no more of his history than you know already, and I shall not attempt the task. What is known leaves him very imperfectly disclosed, but it suffices to show him strong and self-reliant, a man of solitary habits of thought and of untiring industry, and I am content with such a delineation of the man of whom I am about to speak.

Nor do I feel more strongly drawn in the direction of mere eulogy. How greatly I admire Napier and how desirous I am that his merits and his labours should be appreciated will sufficiently appear from my address. But his works will themselves win their due praise and would not, I think, be aided in so doing by my indulging in laudatory epithets. In this connection, however, I would make one remark at the outset of an address dealing well-nigh exclusively with the marvellous invention of logarithms which has immortalised the name of Napier. It must not be thought that it is the sole proof of his greatness that he has left to us. It is true that his other work, taken as a whole, has been long since left behind by the growth of knowledge since his day. In two matters, however, he has claims to our continued gratitude. No mathematician can afford to think lightly of the man who was the first to discover and treat with respect those doubtful entities which are so dear to our souls, *i.e.* impossible quantities;—while he has laid the world of practical life under eternal obligation to him by his introduction of the decimal point and the examples he gave of its use. In his works we find the theory and the practice of decimal fractions as firmly established, and as well understood, as they are at the present day throughout the whole civilised world.[1]

[1] Appendix, p. 27. *Constructio.*

Rejecting then the temptation to indulge in personal history or in mere eulogy, I have had much perplexity in deciding on the nature of my address to you to-day. What new contribution can I bring either to the knowledge of Napier's work or to the appreciation of its merits, when so many skilled mathematicians have already worked at the subject and made full utterances thereon? What qualification have I which entitles me to claim afresh your attention to this well-worked theme?

There is one aspect in which it has occurred to me that I may perhaps find something to contribute, even though it be but the widow's mite. The invention of logarithms came on the world as a bolt from the blue. No previous work had led up to it, nothing had foreshadowed it or heralded its arrival. It stands isolated, breaking in upon human thought abruptly without borrowing from the work of other intellects or following known lines of mathematical thought. It reminds me of those islands in the ocean which rise up suddenly from great depths and which stand solitary with deep water close around all their shores. In such cases we may believe that some cataclysm has thrust them up suddenly with earth-rending force. But can it be so with human thought? Did this discovery come as a revelation to Napier, bursting on him as a light from Heaven, or was it the result of slow growth, the evidences of which are now obliterated, like those rocks whose abrupt sides are due, not to sudden and isolated disruption, but to the denudation which has carried away the neighbouring rocks, which, while they remained, testified to the gradual upheaval of the whole?

The undoubted fact that Napier worked for some twenty years at the invention of logarithms before he published his first book relating to them is, to my mind, decisive upon this point. It must have been a slow and gradual evolution, even though that which remains furnishes so few traces of the earlier efforts. Is it then possible, out of what he has left us and out of the circumstances of the times, to read the history of this evolution, to reconstitute the process of discovery by deciphering the half-effaced records of its growth? It is here that I think I may find something to interest you, and at all events I shall find the subject congenial. I think I may claim wide experience in what may perhaps be termed the infancy of ideas. In the course of my career at the Bar I had great opportunities of studying inventors and their inventions. I learnt, on the one hand, that pioneer inventions are rarely made in the way in which we should have supposed they had been. The man groping in the dark feels his way by devious paths, often guided by trivial and immaterial indications which he has not yet learnt safely to disregard. It is often a fortunate thing that he is willing to follow such guidance because, though it may lead him by circuitous paths, it gives unity of design and consistency to his efforts and keeps them from becoming casual and disconnected essays the success of any one of which must be purely a matter of chance. He must be content though the road be long and weary, for it is only after the light has been reached that he can hope to see what would have been the direct path. And, on the other hand, I learnt how difficult it is, when once the day has broken, to put oneself back into the twilight land and see things with the hazy and shapeless outlines which they then possessed and which made recognition then so impossible. All this experience has, I trust, prepared me to some degree for the task of tracking out the probable course of a discovery where the traces are not too absolutely obliterated, and on this occasion I have used what skill I have thus gained from the past with the aim of discovering how and by what steps Napier advanced from the darkest night which existed when he first took up the problem to the full daylight in which he finally gave it to the world.

All that Napier has left us on the subject of logarithms is contained in two short books, the one known as the *Descriptio*, published in 1614, and the other known as the *Constructio*, published after his death in 1619. Internal evidence as well as the distinct statement of his son, who published the *Constructio*, make it clear that it was in fact written many years before the *Descriptio*, and it represents in many passages an earlier stratum of thought. Both these works were written in Latin—the common tongue of the learned world of his time; but Napier saw and approved of a translation into English of the *Descriptio*, and about twenty-five years ago an excellent translation of the *Constructio* was published in Edinburgh. It is to these translations that I shall refer. The *Constructio* is the fuller and more valuable of the two publications. It is evident that in the *Descriptio* the author published only so much of the reasoning on which his calculations rested as was necessary to enable the mathematical world to appreciate the nature and use of the tables which are to be found there. Indeed, we find Napier expressly stating in it that he does not propose to publish to the world the manner in which the tables were calculated until he finds that they have justified their existence by their acknowledged usefulness.[1] The *Descriptio* therefore bears evidence of being written all at one time, to serve as an introduction and guide to the tables which were printed with it. But the *Constructio* was evidently written at several different times. The order of its contents is peculiar, and there are to be seen in it evidences of different stages of the discovery. Its object was to explain fully the mode in which he had calculated the Tables and incidentally the reasoning on which they were based, but there are no historical references to the way in which he originally arrived either at the idea of a Table of Logarithms or at the method of constructing it.

The first question one asks oneself is, What set Napier

[1] Appendix, *Descriptio*, Plate v.

to work on creating tables which were to enable multiplication to be performed by a process of addition? What first gave him the idea of any such thing? It is very difficult to answer this question. I know of no antecedent work which would suggest it. But there is a peculiarity in the form of his investigations which gives us a useful clue. He usually frames his propositions as though they applied exclusively or at all events specially to sines. Now it is evident that all that concerns logarithms must relate to numbers generally, and that their being sines has no bearing on the matter. Hence his confining his work to sines must indicate that he set out with the idea of working on them only, and that it was only at a later stage and perhaps incidentally that he realised that his results could with like advantage be applied to numbers generally. I conclude from this that his original idea was only to construct tables that would enable the product of two sines to be readily ascertained. If I am right in this, the suggestion may well have come to him from his familiarity with the well-known trigonometrical formula :—

$$\text{Sin A}.\text{Sin B}=\tfrac{1}{2}\{\text{Cos (A}-\text{B)}-\text{Cos (A}+\text{B)}\},$$

which expresses the product of two sines in terms of the cosines of the sum and difference of the angles. Napier, seeing that the existing trigonometrical tables enabled the product of two sines to be found without actual multiplication, may well have conceived the idea of constructing special tables which would do this in a better and more expeditious manner. In no other way can I conceive that the man to whom so bold an idea occurred should have so needlessly and so aimlessly restricted himself to sines in his work, instead of regarding it as applicable to numbers generally. It is true that he often speaks of numbers as an alternative of sines, but one has only to look at his methods of calculation and the general style of his language to see that originally he spoke and thought of sines alone and that the other words indicate later additions.

Suppose then Napier to have started on a scheme for making tables which would give the product of sines without actual multiplication, what was there to guide him to a solution of the problem? There was practically nothing in contemporary or past mathematical knowledge to render him substantial assistance. No doubt many mathematicians had noticed that in a geometrical progression commencing with unity the product of two terms, or the quotient of one term divided by another, would also be a term of the series. Counting unity as the 'zero-term' the product of the 5th and 7th terms would be the 12th, and the quotient of the 7th term divided by the 5th would be the 2nd. This had been a commonplace in mathematics since the days of Archimedes, who states it with great clearness in his *Arenarius*. But there is no reason to think that Napier got it from this source. It must have been noticed independently by many a mathematician with much less ability than Napier possessed. But be that as it may—with the exception of this and of the slightly modified form of the proposition which applies to geometrical progressions not commencing with unity I know of nothing in the knowledge of the world of that time which bore in any way on Napier's task or could give him any aid.

So far as I can read the riddle from what he has left us, the course of his discovery divides itself into three stages. His first step was to realise that in order to create tables which would enable numbers to be multiplied together without actually performing the operation, they must not be represented as resulting from continuous addition as they are in Arabic or Roman notation. They must be represented as resulting from continued multiplication. If you could represent all the numbers that you wanted to use as being powers of one and the same number, then to multiply them together you would only have to add the indices of those powers. But the notion must have seemed an uninviting one from the point of view of practical use. If you pass from one number to the next by multiplication by some uniform factor, the gaps between your numbers will become larger and larger, and the possibility of your being able to represent in that way all the numbers you may want to use must be very remote. It was here that the accident that in his earliest work Napier thought only of sines appears to me to have guided him into the right path. Sines considered as ratios are less than unity, and are now represented with great accuracy by decimal fractions extending to many places of decimals. But at that date decimal fractions were unknown to the world, and it was customary to regard sines as lines drawn in a circle of suitable radius, and the necessary accuracy was obtained by making that radius very large, say 10,000,000. The value of a sine could never exceed this figure, but it might vary from a number almost equal to the radius down to a number however small. Hence it came to pass that Napier's first idea was to start from this large figure of 10,000,000 and to multiply repeatedly by a factor slightly less than 1, choosing that factor so that the first difference would be unity, and therefore all later differences would be smaller than unity. This would ensure his getting a series of numbers expressible as derived by the repeated use of one and the same multiplier, those numbers being so close to one another that one or more must come in between every two consecutive numbers provided those numbers were less than 10,000,000. In this way he hoped to be able to get numbers of the necessary form which would represent all sines to the same degree of approximation as the existing tables of natural sines.

Originally, therefore, Napier contemplated nothing further than repeated multiplications by a factor very close to unity, so that each multiplication reduced very slightly the number operated upon. He started with the number chosen for the radius, namely, 10,000,000, and the factor which he proposed to use was $1-\frac{1}{10,000,000}$. It is obvious that this would at first reduce the radius by unity only, and thereafter the reduction produced by each repeated operation would be yet smaller. If the results of such an operation repeated sufficiently often could have been calculated, it would have given rise to two series of numbers each member of the one having a member corresponding to it in the other. The first series would have consisted of the natural numbers 0, 1, 2, 3, . . ., and would have indicated the number of times the continued multiplication by the chosen factor had been performed. The second series would have consisted of numbers beginning with the radius and decreasing in geometrical progression, and would have indicated the results of performing upon the radius the continued multiplication by the chosen factor 0, 1, 2, 3 . . . times. These latter numbers would be so close to one another as never to differ by more than unity, and in the later stages to differ from one another by much less than unity. In this way every number less than 10,000,000, that is to say every sine, would be very close to some number produced by a known number of these repeated operations of multiplication by the chosen factor, and might without appreciable error be taken to be the actual result of that number of operations. Let us call that number the logarithm. We may then say that the logarithm corresponding to a given sine shows the number of operations of multiplication by the common factor required to arrive at it from the radius, and thus all sines (*i.e.* all numbers less than 10,000,000) are expressed as terms of one and the same geometrical series (of which the first term is the radius 10,000,000), and the logarithm tells us the number of that term.

Not much remains of this early stage of the conception, but traces of it are not absent, and I have no doubt that

Napier's mind passed through it in the beginning. Oddly enough, the word logarithm preserves it, for I think there can be no doubt that logarithm comes from λόγων ἀριθμός, and signifies 'the number of the ratios.' It recalls, therefore, the fundamental notion with which Napier started on his great work. I find also that at the commencement of the *Constructio*, when he is describing the logarithmic table, he says, 'It is picked out from numbers progressing in continuous proportion,'[1] which accurately describes this early conception of the process of constructing logarithmic tables. So far as actual calculation is concerned, Table 1,[2] in which Napier performs the operation a hundred times successively, starting with the radius and making use of the factor $1-\frac{1}{10,000,000}$, is probably a relic of his early work in this direction, though its subsequent use was for a far higher purpose.

It is well to pause here, at the end of the first stage of Napier's discovery, and examine how far he had brought it. A moment's thought will make it clear that his plan would have expressed in powers of the chosen factor the ratio of the sine to the radius, *i.e.* the sine considered as a trigonometrical ratio as we now regard it. Inasmuch as the chosen factor is less than unity (although very near to it), high powers correspond to small sines and *vice versa*. The index of the power of the chosen factor which is equal to a given sine he would have called the logarithm of that sine.

Napier must very soon have realised that, complete as was this conception, the labour of carrying it out was prohibitive. The operations in themselves were singularly easy, and were chosen for that purpose; but when one considers that it would have taken nearly 7,000,000 operations to do one-half of the necessary work, it is obvious that in its unmodified form the scheme was impracticable. But I

[1] Appendix, p. 26. *Constructio.* [2] Appendix, p. 28. *Constructio.*

think we may ascribe to this early stage the predominance in Napier's mind of the principle which is often referred to by him, viz. that in dealing with large numbers we need not trouble about fractions, or as he expresses it in the early part of the *Descriptio*, 'For in great numbers there ariseth no sensible error by neglecting the fragments or parts of an unite.'[1]

The form of the geometrical progression chosen by him for his purpose was, we learn from the *Constructio*, chosen for the ease with which it could be calculated. He says, 'Those geometrical progressions alone are carried on easily which arise by subtraction of an easy part of the number from the whole number.'[2] And he goes on to explain that he means by that the $\frac{1}{1,000,000}$ or $\frac{1}{10,000,000}$, which permits the operation to be done by simply shifting the position of the figures in the subtrahend so many places to the right.

It was the repeated performance of this operation which, in my opinion, led him to the second stage of the discovery. He found himself repeatedly deducting from a number say its ten-millionth part, and thereby the notion of continued multiplication by a factor gradually gave way to the idea of repeatedly taking away one and the same aliquot part from the number arrived at by the preceding operation. This naturally led him to pass in his thought from figures representing quantities arithmetically to the geometrical representation of a quantity by a line, so that his repeated operations were perfectly represented by repeatedly cutting off one and the same fraction of the diminished length of the line operated on. This seems to us a trivial and obvious step, but, in my experience, steps like these often bring great consequences. The value of a step in a process of reasoning may be due, not to the advance actually made thereby, but to the fresh suggestions and novel associations which the thought in its new form brings to the mind. The

[1] Appendix, *Descriptio*, Plate III. [2] Appendix, p. 27. *Constructio.*

identity of the two processes of multiplying by a factor less than unity and cutting off a proportional part is obvious, but the associations which they bring with them are widely different. Figures and lines are both representations of quantity, each with its special advantages and defects. The arithmetical results of such repeated operations are necessarily complex, and the very accuracy which this form of expression renders possible may hamper the mind in making generalisations. But this complexity of results vanishes as soon as a line is chosen in the place of a number. It is a continuous quantity represented and not measured. Still more important in the present case was the consequence that the result of the repeated operations presented itself in the form of the movement of a point along a line, and brought with it all the associations of change of distance by motion.

I have no doubt that very shortly after commencing to represent the results of his repeated operations on the one line, Napier introduced the second line, which was to be the line of logarithms. But its original purpose was a humble one. It had merely to count the operations. The distance which corresponded to each operation was a fixed quantity, so that the length measured on the logarithmic line expressed the number of times the operation had been performed. This is precisely represented by the diagrams which are to be found in the description of a logarithm in the *Descriptio*,[1] and thus understood this length merits the name of logarithm, which is applied to it because it is indeed the number of the ratios.

I have said that the associations which surround the geometrical representation of the process of repeatedly cutting off from a line one and the same aliquot part are wholly different from those that surround the arithmetical process of repeatedly multiplying by a factor less than unity. The successive points along the line, each distant from the one

[1] Appendix, *Descriptio*, Plate III.

next below it by a distance which is a fixed proportion of the length which it marks, naturally gave rise to the idea of a moving point whose velocity is proportional to its distance from the other end of the line. But it is clear that in this stage of the discovery Napier conceived of the point moving with that velocity for a 'certain determinate moment of time.'[1] So long as this restriction was observed the geometrical representation corresponded precisely to the arithmetical operation which he intended it to represent. During each of the same 'determinate moments of time' the point on the logarithm line was moving through equal spaces, so that the 'number measuring it'[1] (*i.e.* the total distance travelled) was truly called the logarithm. Hence we have not as yet got far from the original conception. We have done little more than introduce a geometrical representation of the original arithmetical operations.

But here Napier took a simple and obvious step which ultimately proved of momentous importance. He establishes the proposition that the logarithms of proportionals are 'equally differing.' This is in his eyes merely passing from a single operation to equal groups of that operation. The operation repeated a certain number of times will have the effect of reducing quantities in one and the same proportion whatever the quantities be. But though this generalisation seems obvious, it goes so directly to the very nature and constitution of logarithms that in this single proposition lies the necessary and sufficient test, whether a system of artificial numbers corresponding to the natural numbers is or is not a table of logarithms. If the numbers that correspond to proportionals are equally distant, it must be a table of logarithms. Otherwise, it cannot be.

Napier felt fully the importance of this proposition,—which he expresses in the *Constructio* in the form 'The logarithms of similarly proportioned sines are equidifferent,'[1]

[1] Appendix, p. 30. *Constructio.*

and he literally revels in it. He gives repeated examples of its use—shows how it enables one to find continued and mean proportionals of all kinds, to extract roots, to calculate powers.[1] The triumphs which he ascribes to his invention are all instances of its application. But what is more important for our purposes, it is clear that from the time that he formulated the principle he thought less and less of the individual operations of reduction, and more and more of the effect of those operations when taken in equal groups. His representation of the single operations by equal distances on the logarithmic line adapted itself perfectly to the new line of thought. Equal groups were represented by equal lengths on the logarithmic line, and thus he came to view the addition of a certain length to the logarithms of numbers as giving you the logarithms of those numbers after they had been reduced in a certain proportion.

This concentration of Napier's thoughts on groups of operations identified by equal lengths on the logarithmic line, prepared him for what I view as the third and most interesting stage of his discovery. So long as the effect of the group in the reduction it produced was the same, what mattered it whether it was made out of a larger number of small reductions or a smaller number of large ones? In both cases his principle that 'the logarithms of proportional quantities are equidifferent' would apply. It would only mean that the 'determinate moments' during which the point kept its velocity would be shortened, and the changes of velocity would come more frequently. From this he—no doubt gradually—passed into the stage of contemplating these changes as taking place so frequently that it might be said that at each instant the moving point possessed the exact velocity that it should have were it starting to move for a 'determinate moment,' *i.e.* that its velocity was proportional to its distance from the farther end of the line.

[1] Appendix, *Descriptio*, Plate v.

In effect this is to make the reduction by each single operation infinitely small, and the number of operations infinitely great. But Napier's mind was not of a kind to be troubled by such difficulties. He had marked out for himself a task where theoretical inaccuracies which had no practical effect were religiously to be disregarded, and he must have realised that here there was not even reason to think that there existed a theoretical inaccuracy. However minute the intervals between the changes of velocity, the law held good. What reason to doubt that it would hold good in the ultimate form of the conception, viz. that of a motion with continuously changing velocity?

It is interesting to mark how Napier treats this jump from discontinuity to continuous motion. He does not pass, as Newton did in his *Fluxions*, through a course of reasoning as to infinitesimal quantities. There is no reference to them in Napier's work. He proves his principle by reference to discontinuous motion, and calmly uses it as applicable to continuous motion. But he gives evidence of a certain twinge of conscience in so doing, for after demonstrating his proposition by appeal to repeated operations, *i.e.* to discontinuous motion, he gives us this cryptic sentence:

> And so in all proportionals.
>
> For what affections and symtomes the *Logarithmes* have gotten in their first beginning and generation, the same must they needes retain and keepe afterwards.
>
> But in their beginning and generation they are indued with this affection, and this law is prescribed unto them, that they bee equally differing, when their sines or quantities are proportionall (as it appeareth by the definition of a *Logarithme*, and of both motions, and shall hereafter more fully appeare in the making of the *Logarithmes*). Therefore the *Logarithmes* of proportional quantities are equally differing.[1]

What a sentence for a mathematician to write! It is

[1] Appendix, *Descriptio*, Plate IV.

worthy of a philosopher discussing the existence of the Absolute! It means nothing more than that Napier saw that his work must be true of continuous motion if it was true of all discontinuous motion, and that he was not going to be delayed in his great and practical task by any metaphysical difficulties that he foresaw could not affect his results.

But what a revolution this apparently slight change has effected. All idea of individual operations has disappeared. No arithmetical parallel to the geometrical representation any longer exists. The logarithmic line no longer serves to count operations; it now measures time continuously, it does not count it by ' determinate moments.' How will he adapt this new geometrical conception to arithmetical work when he has no arithmetical equivalent of the process it represents?

Here steps in the faithful servant, the proposition that ' the logarithms of proportionals are equidifferent.' The addition of any the same quantity to a logarithm has still the property of making it correspond to the number arrived at by reducing the number which it originally represented in a certain definite ratio, that ratio being determined by the amount of the addition made to the logarithm. The converse is also true. If numbers are in continued proportion, as are the terms of a geometrical progression, their logarithms differ by a constant quantity. When therefore the logarithm of the common ratio and the logarithm of any one term of the series are known, the logarithm of any other term of the series can readily be ascertained.

But this does not represent all that survived the change from discontinuity to continuous motion. The new conception gave to Napier a ready method of making adjustments in logarithms to correspond to small changes in the numbers they represent. Imagine two numbers differing by a very minute quantity. They will correspond to two points very near to each other on the line which represents the natural numbers. The time occupied by the moving point in passing over the small interval between them will be greater than if it moved throughout at the velocity which it possesses at the commencement of its passage, and less than if it moved throughout at the velocity which it possesses at the end. This follows at once from the fact that the velocity continuously decreases. But the time so taken by the moving point is the difference between the logarithms of the two numbers. By this proposition the difference of the logarithms is fixed within limits which are readily calculated and are so close the one to the other as to enable the actual difference to be fixed with an accuracy more than sufficient for practical purposes if the interval be small.[1]

These two propositions represent all that Napier has saved from the total wreck of his arithmetical methods and conceptions due to the change from the discontinuity which represented individual operations to the continuous change which he could conceive in its geometrical representation, but which had no arithmetical equivalent. It is with these alone that he has to return to his task of calculating arithmetically a table of the logarithms of natural sines.

But they are ample for his purpose. He takes first the series of calculations (which he had no doubt made at the very beginning of his labours) in which beginning from the radius of 10,000,000 he consecutively deducts one ten-millionth part from it and from the numbers so formed. This constitutes a geometrical progression. The terms being in continued proportion, their logarithms are equally differing. This difference is so small that he can fix it by his methods of approximation, and since the first term, the radius, has 0 for its logarithm, he knows the logarithms of all the terms. He chooses the hundredth term. It is, of course, a quantity bristling with decimals, but it is very nearly the same as though 100 had been deducted from the radius. The

[1] Appendix, p. 31. *Constructio.*

successive reductions have not yet got perceptibly smaller than unity. By his methods of adjustment he can allow for the small difference, and thus he gets the logarithm of the radius less 100 exactly.[1] This is his first conquest.

He now creates another geometrical progression, again starting from the radius. The first reduction is now 100 and not unity. He now knows the logarithm corresponding to such reduction, and hence he knows the logarithm of every term of this series. The fiftieth term of this series is, as before, a complicated number, but the difference between it and the radius is very nearly fifty times the 100 which was taken off at the first step. As before, he makes an adjustment which gives him the logarithm of the number which differs from the radius by exactly 5000.[1] With this he starts again, and makes a geometrical series of which this is the initial reduction. This time he does not push the series beyond twenty terms, and by proceeding as before he gets the logarithm which represents the reduction from the radius to a number less than it by exactly 100,000, *i.e.* a reduction from 100 to 99.

He is now ready to commence calculating his tables. He takes the radius and forms from it a geometrical series in which the reduction between consecutive terms is one-hundredth. Sixty-nine terms of such a series are sufficient for his purpose, because he does not desire to go to numbers less than half the radius.[1] By his previous calculations he knows the logarithms of all these terms, and he writes each of them over against the number to which it corresponds. These numbers are widely separated by intervals commencing with 100,000 and diminishing as they proceed. He then takes each of these numbers as the first term of a geometrical series of twenty terms, where the reduction is 5000 out of the 10,000,000, *i.e.* one two-thousandth. He knows the

[1] Appendix, p. 31. *Constructio.*

logarithms of all the terms of these series, because he knows the logarithms of the first terms and the logarithm which corresponds to the common ratio. Thus he has some fourteen hundred numbers, fairly well distributed over the field, of all of which he knows the logarithms. It is true that they are not numbers possessing any interest to him, but that does not matter. They are only to serve as his measuring posts. He now takes the table of natural sines which gives the numbers of which he wishes to calculate the logarithms. Taking each sine separately he examines where it, regarded as a number, comes in the scale. It cannot be far from a measuring post. His methods enable him to make a proper allowance in its logarithm for this small difference in fact, and as the logarithm of the measuring post is known, the logarithm of the sine is known also. It was thus that Napier calculated with infinite labour and pains the thousands of figures that he gave to the world in the Table of Logarithms contained in the *Descriptio*.

I have now given you, as I read it, the line of discovery which led up to Napier's Table of Logarithms which was published in the *Descriptio*. That which impresses me most deeply is his tenacity of aim, combined with his receptivity of new ideas in attaining it. From first to last it was a Table of Logarithms of sines that he proposed to make, and he did not permit himself to be turned aside from that purpose till it was accomplished. His concepts evidently widened as he proceeded, and he must have been sorely tempted to turn from his comparatively restricted task to larger schemes. But he wisely resisted the temptation. He saw that he must create an actual table and give it to the world, or his task was unperformed. Would that other inventors had been equally wise! One of the sad memories of my life is a visit to the celebrated mathematician and inventor, Mr Babbage. He was far advanced in age, but his mind was still as vigorous as ever. He took me through his work-rooms. In the first room I saw the parts of the original Calculating Machine, which had been shown in an incomplete state many years before and had even been put to some use. I asked him about its present form. 'I have not finished it because in working at it I came on the idea of my Analytical Machine, which would do all that it was capable of doing and much more. Indeed, the idea was so much simpler that it would have taken more work to complete the Calculating Machine than to design and construct the other in its entirety, so I turned my attention to the Analytical Machine.' After a few minutes' talk we went into the next work-room, where he showed and explained to me the working of the elements of the Analytical Machine. I asked if I could see it. 'I have never completed it,' he said, 'because I hit upon an idea of doing the same thing by a different and far more effective method, and this rendered it useless to proceed on the old lines.' Then we went into the third room. There lay scattered bits of mechanism, but I saw no trace of any working machine. Very cautiously I approached the subject, and received the dreaded answer, 'It is not constructed yet, but I am working at it, and it will take less time to construct it altogether than it would have taken to complete the Analytical Machine from the stage in which I left it.' I took leave of the old man with a heavy heart. When he died a few years later, not only had he constructed no machine, but the verdict of a jury of kind and sympathetic scientific men who were deputed to pronounce upon what he had left behind him, either in papers or mechanism, was that everything was too incomplete to be capable of being put to any useful purpose.

But so soon as the discovery had actually seen the light in completed form, and its claim to practical usefulness had been enthusiastically allowed by all who learned of it, Napier proceeded most justifiably to destroy the scaffolding which had been so serviceable in the erection of the building. For example, the plan of taking the radius as the starting-point throughout had, I can see, been of inestimable service both in originating his conceptions and in keeping up the continuity of his methods. Accordingly he never varied from it. But before his tables were published he had seen that it was unnecessary thus to start from the radius as that which had zero for its logarithm, and he proclaimed it to be so in the *Descriptio*.[1] We know that at this time he had realised that it would be better to start from unity as the number whose logarithm should be zero. This was consequential on his widened conceptions of logarithms, which connected them with numbers generally and not specially with sines. Such a change of view could create no difficulty when once the Table had been constructed. It merely amounted to subtracting the same quantity from all the logarithms. But its introduction into his conceptions prior to the practical completion of the work might have seriously imperilled its success by interfering with the simplicity of the conceptions which then guided him.

A still more remarkable change which he himself suggested was to follow up this last proposal by fixing unity as the logarithm of 10. That this could be safely done could not possibly have been seen by him until the theoretical basis of his work was complete. It is an example of the truth that from the top of the mountain one can often see how the climb might have been made easier by deviations which to the climbers might well seem to be courting unnecessary difficulties. We now see that this change amounts to making the point on the logarithmic line start with a velocity less in a certain ratio than the point on the line which marks the natural numbers. To have introduced this

[1] Appendix, *Descriptio*, Plate IV.

at an earlier stage would have led to inextricable confusion, and would have complicated beyond measure his arithmetical task. The numerical value and the meaning of this ratio could not have been known except from the completed work. But once the task was completed, Napier was the first to show how the result could be bettered and Tables of Logarithms constructed which would possess much wider utility.[1] He thus has the honour of being the discoverer of logarithms in the two forms in which they are known to and used by the mathematician and by the practical computer respectively.

We all know now that the combination of these two changes, *i.e.* putting zero as the logarithm of unity and unity as the logarithm of 10, has the effect of making 10 the base of the system, so that the logarithm of a number is the index of the power of 10, which is equal to the given number. But this involves the theory of fractional indices, and they were not thought of till many years after the death of Napier. Yet he has left us a remarkable theorem[2] which, in effect, anticipates all that was subsequently discovered in this connection. It is to the following effect:

> The logarithm of any given number is the number of places or figures which are contained in the result obtained by raising the given number to the 10,000,000,000th power,

and he actually finds the logarithm of 2 by this method. To understand the proposition we must remember that the logarithms which he speaks of and publishes in his tables (whether in the original or the later form) are in all cases expressed as large whole numbers, because decimals were not known. Take, for instance, the logarithms which he publishes in the *Descriptio.* They contained seven figures—in other words, they were the true value of the logarithm of the

[1] Appendix, *Descriptio,* Plate IV; Appendix, p. 31. *Constructio.* See also an 'Admonition' printed in Chapter IV of Wright's translation of the *Descriptio.*

[2] Appendix, p. 32. *Constructio.*

number multiplied by 10,000,000 or—which is the same thing—the logarithm of the 10,000,000th power of the number. In the paragraph from which I have thus quoted he is dealing with Tables of Logarithms to the base 10, and these he proposes shall be calculated to ten figures, so that the logarithm in the Table would be the logarithm of the 10,000,000,000th power of the number. His proposition is therefore in truth identical with the statement that the logarithm of a number is the power to which 10 must be raised to equal that number. So soon as decimals and fractional indices became known, this proposition would have immediately led to the relation between a number and its logarithm to the base 10 as we now know it. One could wish for no stronger example of Napier's great insight into mathematical truths that were as yet unknown to the men of his time whenever they bore upon the subject of his long years of thought and labour.

I have now to the best of my ability finished the task I set before me—to trace out the path by which Napier arrived at his epoch-making discovery. If you have understood me aright, my aim has not been to diminish the greatness of his achievement or to make it less wonderful. I have only sought to make it less mysterious. As it stands in his written work it is well-nigh uncanny. Both the books that relate to it were written after he had attained his full knowledge, and any traces of his early efforts that are to be found in them were left there, so to speak, by accident. Hence the whole discovery bursts upon us as though it were due to magic. It requires close examination by a mathematician to connect his conceptions and methods with anything of his date or since, or even with logarithms as they are now conceived of by us. It may well be that to some minds there is a charm in the idea of a discovery leaping in full perfection from the mind of its author without previous imperfect attempts or laborious preparation. For my own part, I prefer to think of his work as more human, as the persistent effort of a great mind to perform a task which it has deliberately set to itself, and which, step by step, it pursues to the end. At the outset it required mighty faith to believe that it was a possible task, that it was not a vain chimera. When that fear had been dispersed, it must have seemed an impracticable task, too great for mortal achievement. But Napier allowed himself to take nothing from his difficulties, except further insight into the subject and suggestions of new devices whereby those difficulties could be overcome or evaded. It is as thus working that I like to think of him. The fascination of reading of the first climbing of a mountain is the tale of how the climbers seized upon all the little helps that the moment afforded until they at length reached the summit. The demonstration that there exists some other easy and comfortable path may merit more gratitude, but it excites less interest. Napier devoted twenty years to this work, many of which—probably the greater part of which—were spent in arriving at the method of achieving it. To my mind it would be sad to think that most of this was, so to speak, wasted because the solution came by a lucky chance at the last. In my view, all these years did their share, and I have tried to show how gradual and continuous was his progress. As to the greatness of the achievement it is needless to speak. Logarithms have played well-nigh as important a part in Mathematical Theory as in practical work. We know infinitely more of their nature and relations than Napier or any man of Napier's age could have done. We have means of calculating them so effective that if all the logarithmic tables in the world were destroyed, the replacement of them would be the work of a few months. But not all the three centuries that have elapsed have added one iota to the completeness or the scope of the two and only existing systems of logarithms as they were left by the genius of John Napier of Merchiston.

APPENDIX

The object of the Appendix is to reproduce such parts of the *Descriptio* and the *Constructio* as are necessary to enable the reader to follow out Napier's method of constructing logarithms and to appreciate the reasoning upon which it is based.

Of the *Descriptio* the title-page and the first and second chapters of the First Book are reproduced as Plates I to VI in facsimile from the translation by Edward Wright, which was made in Napier's lifetime, and submitted to and approved by him. These two chapters contain all that is requisite for the above purpose. The remainder of the First Book consists of a description of the Tables printed in the *Descriptio*, and explanations and examples of the method of using them. The Tables consist of seven columns giving respectively for each minute in the first half of the quadrant the following quantities:—(1) the angle, (2) its natural sine, (3) the logarithm of its sine, (4) the logarithm of its tangent, (5) the logarithm of its cosine, (6) its natural cosine, (7) the complement of the angle. It will be seen, therefore, that the figures in (4) are obtained by subtracting those in (5) from those in (3). The logarithms are to seven figures.

The quotations from the *Constructio* are taken by permission from the translation by William Rae Macdonald, F.F.A.[1] Their object is to enable the reader to follow Napier's own account of his method of constructing his Table of Logarithms and the reasoning on which it was based. Explanatory remarks have been added where necessary; they are in all cases enclosed in brackets.

Though it is believed that these extracts will be sufficient for their purpose, they very imperfectly represent the *Constructio* as a whole. It will amply repay reading *in extenso*.

[1] *The Construction of the Wonderful Canon of Logarithms by Lord Napier, translated from Latin into English with Notes and a Catalogue of the Various Editions of Napier's Work by William Rae Macdonald, F.F.A.* William Blackwood & Sons, Edinburgh and London, 1889.

The Construction of
The Wonderful Canon of Logarithms;
(herein called by the Author the Artificial Table)
and their relations to their natural Numbers.

1. A Logarithmic Table *is a small table by the use of which we can obtain a knowledge of all geometrical dimensions and motions in space, by a very easy calculation.*

It is deservedly called very small, because it does not exceed in size a table of sines; very easy, because by it all multiplications, divisions and the more difficult extractions of roots are avoided; for by only a very few most easy additions, subtractions and divisions by two, it measures quite generally all figures and motions.

It is picked out from numbers progressing in continuous proportion.

2. *Of continuous progressions, an arithmetical is one which proceeds by equal intervals; a geometrical, one which advances by unequal and proportionally increasing or decreasing intervals.*

Arithmetical progressions: 1, 2, 3, 4, 5, 6, 7, etc.: or 2, 4, 6, 8, 10, 12, 14, 16, etc. Geometrical progressions: 1, 2, 4, 8, 16, 32, 64, etc.: or 243, 81, 27, 9, 3, 1.

3. *In these progressions we require accuracy and ease in working. Accuracy is obtained by taking large numbers for a basis; but large numbers are most easily made from small by adding cyphers.*

Thus instead of 100000, which the less experienced make the greatest sine, the more learned put 10000000, whereby the difference of all sines is better expressed. Wherefore also we use the same for radius and for the greatest of our geometrical proportionals.

4. *In computing tables, these large numbers may again be made still larger by placing a period after the number and adding cyphers.*

Thus in commencing to compute instead of 10000000 we put 10000000·0000000, lest the most minute error should become very large by frequent multiplication.

5. *In numbers distinguished thus by a period in their midst, whatever is written after the period is a fraction, the denominator of which is unity with as many cyphers after it as there are figures after the period.*

Thus 10000000·04 is the same as $10000000\frac{4}{100}$; also 25·803 is the same as $25\frac{803}{1000}$; also 9999998·0005021 is the same as $9999998\frac{5021}{10000000}$ and so of others.

6. *When the tables are computed, the fractions following the period may then be rejected without any sensible error. For in our large numbers, an error which does not exceed unity is insensible and as if it were none.*

Thus in the completed table, instead of 9987643·8213051, which is $9987643\frac{8213051}{10000000}$, we may put 9987643 without sensible error.

7. *Besides this, there is another rule for accuracy; that is to say, when an unknown or incommensurable quantity is included between numerical limits not differing by many units.*

[Here follows a series of propositions dealing very thoroughly with the method of limits. This method of determining the degree of accuracy attained naturally became of the greatest importance as soon as Napier's conceptions became based on continuous change.]

13. *The construction of every arithmetical progression is easy; not so, however, of every geometrical progression.*

This is evident, as an arithmetical progression is very easily formed by addition or subtraction; but a geometrical progression is continued by very difficult multiplications, divisions or extractions of roots.

Those geometrical progressions alone are carried on easily which arise by subtraction of an easy part of the number from the whole number.

14. *We call easy parts of a number, any parts the denominators of which are made up of unity and a number of cyphers, such parts being*

obtained by rejecting as many of the figures at the end of the principal number as there are cyphers in the denominator.

Thus the tenth, hundredth, thousandth, 10000th, 100000th, 10000000th parts are easily obtained because the tenth part of any number is got by deleting its last figure, the hundredth its last two, the thousandth its last three figures and so with the others, by always deleting as many of the figures at the end as there are cyphers in the denominator of the part. Thus the tenth part of 99321 is 9932, its hundredth part is 993, its thousandth 99, etc.

15. *The half, twentieth, two hundredth, and other parts denoted by the number two and cyphers, are also tolerably easily obtained; by rejecting as many of the figures at the end of the principal number as there are cyphers in the denominator, and dividing the remainder by two.*

Thus the 2000th part of the number 9973218045 is 4986609, the 20000th part is 498660.

16. *Hence it follows that if from radius with seven cyphers added you subtract its* 10000000*th part, and from the number thence arising its* 10000000*th part, and so on, a hundred numbers may very easily be continued geometrically in the proportion subsisting between radius and the sine less than it by unity, namely between* 10000000 *and* 9999999*: and this series of proportionals we name the First table.*

[Here follows the calculation of the First table. In it the second term is less than the radius by unity.]

17. *The Second table proceeds from radius with six cyphers added, through fifty other numbers decreasing proportionally in the proportion which is easiest, and as near as possible to that subsisting between the first and last numbers of the First table.*

[Here follows the calculation of the Second table. In it the second term is less than the radius by 100.]

18. *The Third table consists of sixty-nine columns, and in each column are placed twenty-one numbers, proceeding in the proportion which is easiest, and as near as possible to that subsisting between the first and last numbers of the Second table.*

Whence its first column is very easily obtained from radius with five *cyphers added by subtracting its 2000th part, and so from the other numbers as they arise.*

[Here follows the calculation of the first column of the Third table. In it the second term is less than the radius by 5000.]

19. *The first numbers of all the columns must proceed from radius with four cyphers added, in the proportion easiest and nearest to that subsisting betweeen the first and the last numbers of the first column.*

As the first and the last numbers of the first column are 10000000·0000 and 9900473·5780, the easiest proportion very near to this is 100 to 99. Accordingly sixty-eight numbers are to be continued from radius in the ratio of 100 to 99 by subtracting from each one of them its hundredth part.

20. *In the same proportion a progression is to be made from the second number of the first column through the second numbers in all the columns and from the third through the third, and from the fourth through the fourth, and from the others respectively through the others.*

[Here follows the calculation of the remaining columns of the Third table. The ratio in each column is the same as in the first column.]

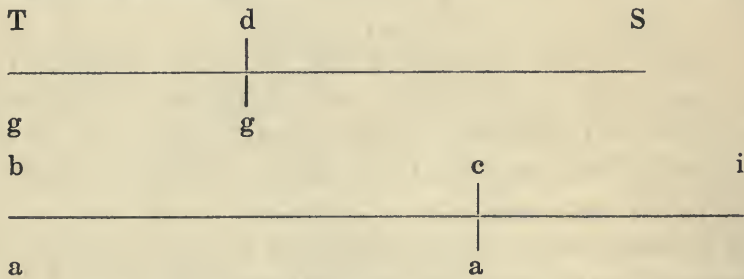
21. *Thus, in the Third table, between radius and half radius, you have sixty-eight numbers interpolated, in the proportion of* 100 *to* 99, *and between each two of these you have twenty numbers interpolated in the proportion of* 10000 *to* 9995; *and again in the Second table between the first two of these, namely, between* 10000000 *and* 9995000, *you have fifty numbers interpolated in the proportion of* 100000 *to* 99999; *and finally, in the First table between the latter, you have a hundred numbers interpolated in the proportion of radius or* 10000000 *to* 9999999; *and since the difference of these is never more than unity, there is no need to divide it more minutely by interpolating means, whence these three tables, after they have been completed, will suffice for computing a Logarithmic Table.*

Hitherto we have explained how we may most easily place in table sines or natural numbers progressing in geometrical proportions.

22. *It remains in the Third table at least, to place beside the sines or natural numbers decreasing geometrically their logarithms or artificial numbers increasing arithmetically.*

[He now proceeds to his definitions of increasing and decreasing arithmetically and geometrically. The language differs from that of the *Descriptio*, Chapter I., but the substance is much the same. This leads up to his definition of a logarithm.]

26. *The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine.*



Let the line TS be radius, and dS a given sine in the same line: let g move geometrically from T to d in certain determinate moments of time. Again, let bi be another line, infinite towards i, along which, from b, let a move arithmetically with the same velocity as g had at first when at T: and from the fixed point b in the direction of i let a advance in just the same moments of time up to the point c. The number measuring the line bc is called the logarithm of the given sine dS.

27. *Whence nothing is the logarithm of radius.*

For, referring to the figure when g is at T making its distance from S radius, the arithmetical point d beginning at b has never proceeded thence. Whence by the definition of distance nothing will be the logarithm of radius.

[He now obtains from this definition the limits of the logarithm of a sine which is very nearly equal to the radius, and thus obtains the logarithm of the second term of the First table, *i.e.* of the ratio of consecutive terms of that series. He then takes up the case of proportionals.]

36. *The logarithms of similarly proportioned sines are equidifferent. This necessarily follows from the definitions of a logarithm and of the two motions. For since by these definitions arithmetical increase always the same corresponds to geometrical decrease similarly pro-*

portioned, of necessity we conclude that equidifferent logarithms and their limits correspond to similarly proportioned sines.

[He now obtains the limits between which the difference of the logarithms of two sines which are near together must lie. This he expresses as follows :—]

39. *The difference of the logarithms of two sines lies between two limits ; the greater limit being to radius as the difference of the sines to the less sine, and the less limit being to radius as the difference of the sines to the greater sine.*

[He applies this to find the logarithms of all the terms of the Second and Third tables. He then uses it to find from these the logarithms of all natural sines included within the limits of the Table. To find the logarithms of those not so included he uses the following theorems :—]

51. *All sines in the proportion of two to one have* 6931469·22 *for the difference of their logarithms.*

52. *All sines in the proportion of ten to one have* 23025842·34 *for the difference of their logarithms.*

[Thus if a sine is less than half the radius he multiplies it by 2 or 10 or powers of those numbers till he gets a number which lies within the limits of the tables and whose logarithm is therefore known. This gives him the logarithm of the original sine. He then describes in detail how to construct a logarithmic table by these means, and ends thus :—]

We have computed this Table to each minute of the quadrant, and we leave the more exact elaboration of it, as well as the emendation of the table of sines, to the learned to whom more leisure may be given.

[There follows an Appendix of great interest in view of the change of the introduction of the later form of logarithms. The more important parts are set out below.]

APPENDIX

On the Construction of another and better kind of Logarithms, namely one in which the Logarithm of unity is 0.

Among the various improvements of Logarithms the more important is that which adopts a cypher as the Logarithm of unity, and 10,000,000,000 *as the Logarithm of either one tenth of unity or ten times unity. Then these being once fixed the Logarithms of all other numbers necessarily follow.*

* * * * *

The Relations of Logarithms & their natural numbers to each other.

1. [A.] *Let two sines and their Logarithms be given. If as many numbers equal to the less sine be multiplied together as there are units in the Logarithm of the greater; and on the other hand, as many numbers equal to the greater sine be multiplied together as there are units in the Logarithm of the less; two equal numbers will be produced, and the Logarithm of the sine so produced will be the product of the two logarithms.*

* * * * *

8. [B.] *If a first sine divide a third as many times successively as there are units in A; and if a second sine divides the same third as many times successively as there are units in B; also if the same first divide a fourth as many times successively as there are units in C; and if the same second divide the same fourth as many times successively as there are units in D: I say that the ratio of A to B is the same as that of C to D, and as that of the Logarithm of the second to the Logarithm of the first.*

9. [C.] *Hence it follows that the Logarithm of any given number is the number of places of figures which are contained in the result obtained by raising the given number to the* 10,000,000,000*th power.*

10. *Also if the index of the power be the Logarithm of ten the number of places less one, in the power or multiple, will be the Logarithm of the root.*

> Suppose it is asked what number is the Logarithm of 2. I reply, the number of places in the result obtained by multiplying together 10,000,000,000 of the number 2.
>
> But, you will say, the number obtained by multiplying together 10,000,000,000 of the number 2 is innumerable. I reply, still the number of places in it, which I seek, is numerable.
>
> Therefore with 2 as the given root, and 10,000,000,000 as the index, seek for the number of places in the multiple, and not for the multiple itself; and by our rule you will find 301029995, etc., to be the number of places sought and the Logarithm of the number 2.

A
DESCRIPTION OF THE ADMIRABLE TABLE OF LOGARITHMES:

WITH

A DECLARATION OF THE MOST PLENTIFVL, EASY, and speedy vse thereof in both kindes of Trigonometrie, as also in all *Mathematicall calculations*.

INVENTED AND PVBLISHED IN LATIN BY THAT Honorable L. IOHN NEPAIR, Baron of *Marchiston*, and translated into English by the late learned and famous Mathematician *Edward Wright*.

With an Addition of an Instrumentall Table to finde the part proportionall, invented by the Translator, and described in the end of the Booke by HENRY BRIGS *Geometry-reader at Gresham-house in London.*

All perused and approued by the Author, & published since the death of the Translator.

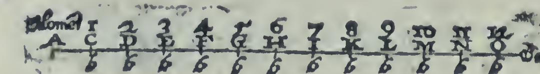
LONDON,
Printed by NICHOLAS OKES.
1616.

A DESCRIPTION OF THE *ADMIRABLE TABLE OF LOGARITHMES,* WITH THE MOST PLENTIFVL, EASIE, AND READY Vse thereof in both kindes of *Trigonometrie, as also in all Mathematicall Accounts.*

THE FIRST BOOKE.

CHAP. I.
Of the Definitions.

A LINE *is said to increase equally, when the poynt describing the same, goeth forward equall spaces, in equall times, or moments.* 1. Definition.



Let A be a poynt, from which a line is to be drawne by the motion of another poynt, which let be B.

Now in the first moment, let B moue from

B A

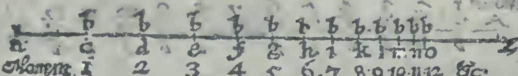
A to C. In the second moment from C to D. In the third moment from D to E, & so forth infinitely, describing the line A C D E F, &c. The spaces A C, C D, D E, E F, &c. And all the rest being equall, and described in equall moments (or times.) This line by the former definition shall be said to increase equally.

A Corellary or consequent. *Therefore by this increasing, quantities equally differing, must needes be produced, in times equally differing.*

As in the Figure before, B went forward from A to C in one moment, and from A to E in three moments. So in sixe moments from A to H: and in 8 moments from A to K. And the differences of those moments, one and three, and of these 6 and 8 are equall, that is to say two.

So also of those quantities A C, and A E, and of these, A H, and A K, the differences C E, and H K are equall, and therefore differing equally, as before.

2. *Definition.* *A Line is said to decrease proportionally into a shorter, when the poynt describing the same in equall times, cutteth off parts continually of the same proportion to the lines from which they are cut off.*



For examples sake. Let the line of the whole sine a Z be to bee diminished proportionally: let the poynt diminishing the same by his motion

motion be b: and let the proportion of each part to the line from wch it is cut off, be as Q R to Q S. Therefore in what proportion Q S is cut in R, in the same proportion (by the 10 of the 6 of *Euclid*) Let a Z be cut in c. and so let b. running from a to c in the first moment, cut off a c from a Z, the line or sine c Z remaining.

And from this c Z let b proceeding in the second moment, cut off the like segment, or part, as Q R to Q S: and let that bee c d, leauing the sine. d Z. From which therefore in the third moment, let b in like manner, cut off the segment d e, the sine e Z being left behinde. From which likewise in the fourth moment, by the motion of b, let the segment e f be cut off, leauing the sine f Z. From this f Z in the fifth moment, let b in the same proportion cut off the segment f g, leauing the sine g Z, and so forth infinitly. I say therfore out of the former definition, that here the line of the whole sine a Z, doth proportionall decrease into the signe g Z, or into any other last sine, in which b stayeth, and so in others.

Hence it followeth that by this decrease in equall moments (or times) there must needes also bee left proportionall lines of the same proportion. *A Corelary.*

For what continuall proportion there is before of the sines to be diminished, a z, c z, d z, e z, f z, g z, h z, i z, and k z, &c. and of the segments cut off from them, a c, c d, d e, e f, f g, g h, h i, and i k, there must needes be also the same proportion of the sines remaining, that is, c z, d z, e z, f z, g z, h z, i z, and k z, as may manifestly appeare by the 19 *Prop.* 5. and 11. *Prop.* 7, *Euclid.*

3 Def. *Surd quantities, or vnexplicable by number, are said to be defined, or expressed by numbers very neere, when they are defined or expressed by great numbers which differ not so much as one vnite from the true value of the Surd quantities.*

As for example. Let the semidiameter, or whole sine be the rational number 10000000 the sine of 45 degrees shall be the square root of 50, 000, 000,000,000, which is surd, or irrationall and inexplicable by any number, & is included between the limits of 7071067 the lesse, and 7071068 the greater: therfore, it differeth not an vnite from either of these. Therefore that surd sine of 45 degrees, is said to be defined and expressed very neere, when it is expressed by the whole numbers, 7071067, or 7071068, not regarding the fractions. For in great numbers there ariseth no sensible error, by neglecting the fragments, or parts of an vnite.

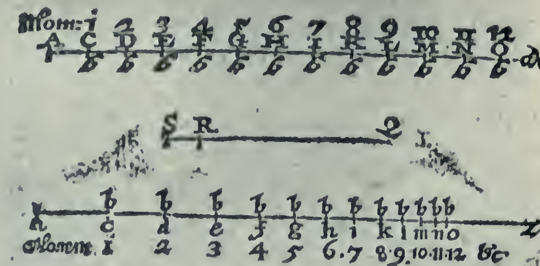
4 Def. *Equall-timed motions are those which are made together, and in the same time.*

As in the figures following, admit that B be moued from A to C, in the same time, wherin *b* is moued from *a* to *c* the right lines A C & *a c*, shall be sayd to be described with an equall-timed motion.

5 Def. *Seeing that there may bee a slower and a swifter motion giuen then any motion, it shall necessarily follow, that there may be a motion giuen of equall swiftnesse to any motion (which wee define to be neither swifter nor slower.)*

6 Def. *The* Logarithme *therfore of any sine is a number very neerely expressing the line, which increased*

sed equally in the meane time, whiles the line of the whole sine decreased proportionally into that sine, both motions being equal-timed, and the beginning equally swift.



As for example. Let the 2 figures going afore bee here repeated, and let B bee moued alwayes, and euery where with equall, or the same swiftnesse wherewith *b* beganne to bee moued in the beginning, when it was in *a*. Then in the first moment let B proceed from A to C, and in the same time let *b* moue proportionally from *a* to *c*, the number defining or expressing A C shal be the *Logarithme* of the line, or sine *c* Z. Then in the second moment let B bee moued forward from C to D. And in the same moment or time let *b* be moued proportionally from *c* to *d*, the number defining A D, shall bee the *Logarithme* of the sine *d* Z. So in the third moment let B go forward equally from D to E, and in the same moment let *b* be moued forward proportionally from *d* to *e*, the number expressing A E the *Logarithme* of the sine *e* Z. Also in the fourth moment, let B proceed

PLATE IV

ceed to F, and *b* to *f*, the number A F shall be the *Logarithme* of the sine *f z*. And keeping the same order continually (according to the former definition) the number of A G shall be the *Logarithme* of the sine *g z*. A H the *Logarithme* of the sine *h z*. A I the *Logarithme* of the sine *i z*. A K the *Logarithme* of the sine *k z*, and so forth infinitely.

A cōsequēt. *Therefore the Logarithme of the whole sine* 1000000 *is nothing, or* 0: *and consequently the Logarithmes of numbers greater then the whole sine, are lesse then nothing.*

For seeing it is manifest by the definition, that the sines decreasing from the whole sine, the *Logarithmes* increase from nothing: therefore contrariwise the numbers which yet we call Sines, increasing vnto the whole sine, that is to 1000000; the *Logarithmes* must needs decrease to 0. or nothing: and by consequent the *Logarithmes* of numbers increasing aboue the whole sine 1000000, which wee call *Secants*, or *Tangents*, and no more sines, shall be lesse then nothing.

Therefore we call the Logarithmes of the sines Abounding, because they are alwayes greater then nothing, and set this marke + before them, or else none. But the Logarithmes which are lesse then nothing, we cal Defectiue, or wanting, setting this marke — before them.

It was indeed left at libertie in the beginning, to attribute nothing, or 0. to any sine or quantitie for his Logarithme: but it was best to fit it to the whole sine, that the Addition or Subtraction of that Logarithme which is most frequent in all Calculations, might neuer after be any trouble to vs.

CHAP.

CHAP. II.

Of the Propositions of Logarithmes.

THE Logarithmes of Proportionall numbers and quantities are equally differing. Propos. 1.

As for example. The Logarithmes of the proportionall sines, namely *c z*, which is to *e z*, as *h z* is to *k z*, are respectiuely the numbers defining A C, A E, A H, A K, (as is manifest by the 6 Definition.) Now A C, and A E differ by the difference C E, and A H and A K by the difference H K. But by the first definition and his Corolarie C E and H K, are equall: therefore the *Logarithmes* of the foresaid proportional sines are equally differing. And so in all proportionals.

For what affections and symtomes the *Logarithmes* haue gotten in their first beginning and generation, the same must they needes retaine and keepe afterwards.

But in their beginning and generation, they are indued with this affection, and this law is prescribed vnto them, that they bee equally differing, when their sines or quantities are proportionall (as it appeareth by the definition of a *Logarithme*, and of both motions, and shall hereafter more fully appeare in the making of the *Logarithmes*.) Therefore the *Logarithmes* of proportional quantities are equally differing.

Of the Logarithmes of three proportionals, the double of the second or meane, made lesse by the first, is equall to the third. Propos. 2.

 Seeing

Seeing that by the first propos. the difference of the *Logarithme* of the first and second, is equall to the difference of the *Logarithmes* of the second and third, that is, the second made lesse by the first, is equall to the third, lesse by the second: therefore the second being added to both sides of the equation twice, the second, or the double of the second made lesse by the first, shall come forth equall to the third, which was to bee proued.

Propos. 3. *Of the Logarithmes of three proportionals, the double of the second, or middle one, is equall to the summe of the extremes.*

By the second Proposition next going before, the double of the second, made lesse by the first, is equall to the third. To both the equall sides adde the first, and there shall arise the double of the second equall to the first and third, that is, to the summe of the extremes, which was to bee demonstrated.

Propos. 4. *Of the Logarithmes of foure proportionals, the summe of the second and third, made lesse by the first, is equall to the fourth.*

Seeing by the first Proposition of the *Logarithmes* of 4 proportionals, the second made lesse by the first, is equall to the fourth lesse by the third: adde the third to both sides of the equality, and the second and third made lesse by the first, shall bee made equall to the fourth, which was propounded.

Propos. 5. *Of the Logarithmes of foure proportionals, the summe of the middle ones, that is, of the second and third, is equall to the Logarithme of the extreames, that is to say, the first and fourth.*

By the 4 proposition going afore the 2 & third

third made lesse by the first, were equall to the fourth: to both sides of the equality adde the first, and the second more by the third shall bee made equall to the fourth, more by the first, which was to be demonstrated.

Propos. 6. *Of the Logarithmes of foure continuall proportionals, the triple of either of the middle ones, is equall to the summe of the further extreame, and the double of the neerer.*

By the second proposition, the double of the second made lesse by the first, is equall to the third; and by the third proposition the double of this, that is, the fourefold of the second made lesse by the double of the first, shall be equall to the summe of his extreames, that is, the fourth more by the second. Now, if from both sides of the equality you subtract the second, the triple of the second made lesse by the double of the first, shall be made equall to the fourth. Againe, to the sides of this equality adde the double of the first, and there shall arise the triple of the second, equall to the fourth, more by the double of the first, which wee vndertooke to proue.

An Admonition.

Hitherto we haue shewed the making and symptomes of *Logarithmes*; Now by what kinde of account or method of calculating they may be had, it should here bee shewed. But because we do here set down the whole Tables, and all his *Logarithmes* with their Sines to euery minute of the quadrant: therefore passing ouer the doctrine of making *Logarithmes*, til a fitter time, we make haste to the vse of them: that the vse and profit of the

 thing

PLATE VI

thing being first conceiued, the rest may please the more, being set forth hereafter, or else displease the lesse, being buried in silence. For I expect the iudgement and censure of learned men hereupon, before the rest rashly published, be exposed to the detraction of the enuious.

Chap. III.

Containing the description of the Table of Logarithmes, and of the seuen Columnes thereof.

1 Section. THE first Columne is expresly of the Arches increasing from 0 to 45 degrees, and is also vnderstood to bee of their remainders to a semicircle.

2 Section. The seuenth columne is of arches decreasing from a quadrant to 45 degrees, and is also vnderstood to bee of their remainders to a semicircle.

3 Section. So the Arches of the one columne are the complements of the Arches of the other answering ouer-against them.

4 And in the first is expressed the lesse sharpe angle of any right-lined right-angled triangle.

5 But in the seuenth ouer against it, is placed the greater sharpe angle of the same right-angled triangle.

6 In the second columne are the sines of the arches of the first columne.

7 And these are the lesse legges subtending the lesse angle of a right angled triangle, whose Base, or Hypotenuse is the whole sine.

8 In the sixth columne are the sines of the arches of the seuenth columne.

9 And

And these are the greater legges subtending 9
the greater sharpe angle of the same right-angled triangle, whose Hypotenuse is the whole sine.

Hence it followeth, that of the whole sine, and 10
the sine of the second columne, and the sine of the sixth columne answering ouer-against the same, there is made a triangle that is equiangled, and like to any right-angled right-lined triangle.

The third columne containeth the Logarithmes 11
of the arches and sines towards the left hand.

Which are the Logarithmes of the proportion 12
of the lesse legge of a right-angled triangle, to the Hypotenuse of the same.

And they are also the Logarithmes of the com- 13
plements of the arches and sines towards the right hand, which we call Antilogarithmes.

The first columne containeth the Logarithmes 14
of the arches and sines towards the right hand.

Which are the Logarithmes of the proportion 15
of the greater legge of a right-angled triangle, to the Hypotenuse of the same.

They are also the Antilogarithmes *of the* 16
arches and sines towards the left hand, or the Logarithmes of the complements.

Lastly, the fourth or middle columne contai- 17
neth the differences betweene the Logarithmes of the third and fifth columnes. And so this columne is two-fold, Abounding and Defectiue.

Those differences are Abounding, which arise 18
out of the subtraction of the Logarithmes of the fifth columne from the Logarithmes of the third columne.

But the differences arising by subtraction of 19
the Logarithmes of the third columne out of the Logarithmes of the fifth columne, are Defectiue, which therefore are lesse then nothing.

The Abounding differences are called the dif- 20
ferentiall

JOHN NAPIER OF MERCHISTON[1]

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Writing about the middle of the eighteenth century, David Hume proclaimed John Napier of Merchiston as 'the person to whom the title of a *great man* is more justly due than to any other whom his country ever produced.' This judgment of Hume is the more remarkable, seeing he was himself naturally disposed to exalt literature above science. Moreover, there had been another Scot, partly a contemporary of Napier, who in the estimation of many of his countrymen and of many in continental circles might have disputed the palm with him. George Buchanan had been described by Milton's antagonist Salmasius as 'the greatest man of his age'; by Grotius as *Scotiae illud numen*; and a stream of testimony since their day had confirmed their opinion. Hume's own ruling passion, as he tells us, was to distinguish himself in letters; when he awarded the first place among his countrymen to Napier, therefore, it was doubtless from an enlightened conviction that his work had been of the greater service to humanity. Both Buchanan and Napier, it may be noted in passing, received one remarkable tribute to the impressiveness of their individualities; both in different ways appealed to the imagination of the people. By reason of his association with James vi and his reputation for caustic and witty sayings, Buchanan, the fastidious

[1] This sketch of Napier's life is mainly based on Mark Napier's *Memoirs of John Napier of Merchiston: His Lineage, Life, and Times* (Edinburgh, 1834), and on entries in the *Privy Council Register of Scotland*.

scholar of the Renaissance, was transformed into a Court fool; and Napier, the explorer of the secrets of nature, passed among his countrymen for a trafficker with Satan.

Even to-day a certain mystery surrounds the figure of the Laird of Merchiston. Appearing at the time he did, and in an environment seemingly so strangely in contrast with his special pursuits, he strikes us as the most singular of apparitions among his contemporaries. He had one fellow-countryman, indeed, as a predecessor in the study of physical science. In the thirteenth century Michael Scott, like his contemporary Roger Bacon, had given his attention to that study; had gained a continental reputation as wide as Napier's, and an equally evil name among his countrymen of being in league with the infernal powers. But between Michael and Napier we can name no Scot whose interest lay specially in the domain of science, and the explanation is simple. In Scotland, as in other countries, the universities were the exclusive centres of intellectual activity, and the studies at the universities were under the sole dominion of the Church, which naturally laid its ban on investigations that might imperil its own teaching. By his isolation Napier is thus wrapped in a certain mystery, and the mystery is deepened by the fact that we know so little of him, and that what we do know is at times strangely incongruous with the main preoccupations of his life.

He came of an ancestry that appears to have had the general characteristics of their age and class, and what is interesting is that Napier, the philosopher, had evidently his fair share of these. The history of the family, which begins with Alexander Napier, a burgess of Edinburgh, in the first half of the fifteenth century, shows that they were a strenuous race, by capacity and energy of character well able to advance their interests and to hold their own throughout a time when force made light of law. Most of them played a more or less important part in the public affairs of the country. The second Napier was Comptroller of the King's Household, and was twice sent abroad on foreign embassies; the third sat in Parliament, and several received the honour of knighthood. Three of them fell in battle: one at Sauchieburn,[1] a second at Flodden, and a third at Pinkie. With their honours they also gained lands, and on the death of his father John succeeded to what was for the time a valuable estate. The original Castle of Merchiston with which the Napiers are associated appears to have been built at some period during the fifteenth century, and, though it underwent much alteration in the succeeding times, it must have been an imposing structure from the first. In Napier's own day Craigmillar Castle and Merchiston Castle were the two strongest places in the neighbourhood of Edinburgh. Its proximity to the Scottish capital gave an importance to the Napier family which is illustrated by a significant fact: during the fifteenth century no fewer than three Napiers were provosts of Edinburgh. As at that period only persons of power and influence in the neighbourhood were chosen to the office, the inference is that the Napiers were in a position to be protectors of the city's interests.

Sir Archibald Napier, the seventh of Merchiston, and the father of John, fully maintained the repute of his ancestors for energy and sagacity. He added to the ancestral domains —specially in the Lennox, with which we shall find his son afterwards associated. In his day, also, he played a notable public part. He was a Justice-Depute under the Earl of Argyll, and for more than thirty years was Master of the Mint—a fact to be remembered as involving his son in an unpleasant experience. What we have specially to note in connection with Sir Archibald, however, is that he identified himself with Protestantism from the first. He sat several times in the General Assemblies of the Reformed Church,

[1] It should be said that it is only a conjecture that the third Napier fell at Sauchieburn.

but apparently his zeal was not such as to satisfy either religious party in the country. When in 1570 civil war broke out between the supporters of Queen Mary and the supporters of her son, his conduct was so dubious that he incurred the hostility of both, and his ancestral castle suffered at the hands of King's men and Queen's men alike. But the fact that he was really a Protestant was to have a decisive influence on the career of his son. The inventor of logarithms inherited the religious opinions of his father, and held them with an intensity that received remarkable expression in a book which brought him greater fame in his own day than his contribution to mathematical science. Sir Archibald was twice married—his first wife, the mother of John, being Janet Bothwell, daughter of Sir Francis Bothwell, and sister of Adam Bothwell, Bishop of Orkney, known to history as an active agent in effecting the unhallowed union of Mary and the Earl of Bothwell. By his first marriage he had three children, of whom John was the eldest; by his second he had ten, with whom John was to have his own troubles.

The 'marvellous Merchiston' (so he was known to the populace of his day) was born at Merchiston Castle in 1550. The period in which his birth and boyhood fell is the most momentous in the national history, and it determined and gave their peculiar character to his fundamental conceptions of human life and destiny. At the date of his birth the controversy had already begun which was eventually to cleave in twain the history of the Scottish people. The issue whether Roman Catholicism or Protestantism was to prevail was already joined. In 1546, four years before Napier was born, George Wishart was condemned by the Church and burned as a heretic, and in the same year Cardinal Beaton, the principal agent in his death, was assassinated. In 1547 John Knox began his mission which, after an interval, he was to see crowned with success. During the first ten years of Napier's life the struggle between the two religions was virtually settled. Between the years 1550 and 1560 the country was distracted by civil war, one party being for the old religion and alliance with France, the other for Protestantism and alliance with England. The contest ended in the victory of the Protestant party, and in 1560 a Convention of the Estates set up Protestantism as the national religion. It is in youth that the strongest and most permanent prepossessions and prejudices are formed, and we may trace the origin of Napier's abiding horror of the Church of Rome to the air which he breathed in the opening years of his life. As we shall see, it came to be his burning conviction that the salvation of mankind was bound up with the overthrow of the Papacy.

Of his early days no record has been preserved, though we know that they were mainly spent at Merchiston Castle. We do not even know where he received his early education. The High School of Edinburgh had existed from 1519, and the sons of barons were in the habit of attending it; but from a letter of his uncle, the Bishop of Orkney, to the elder Napier we are led to infer that John may have been educated at home. 'I pray you, Sir,' the Bishop writes in 1560, 'to send your son John to the schools, either to France or Flanders, for he can learn no good at home, nor get any profit in this most perilous world, that he may be saved in it, that he may do friends [*sic*] after honour and profit, as I doubt not but he will.'

In 1563, the year of his mother's death, John was sent to the University of St. Andrews, the mother university of Scotland. He was only thirteen, but this was the usual age at which lads then entered the universities. Of the three colleges that composed the University, St. Salvator's was chosen for him—possibly because its Head was the most distinguished teacher of his day in Scotland. He was Dr. John Rutherford, who for many reasons was a remarkable person. He had been educated in France, and had taught in one of the most famous schools in that country —the Collège de Guyenne in Bordeaux, where Buchanan had also taught. With Buchanan, also, he had gone on the ill-fated expedition to the lately founded University of Coimbra, in Portugal, the new professors of which were eventually dispersed by the Jesuits—Buchanan, as the most aggressive heretic of the band, being consigned to the dungeon of the Inquisition in Lisbon. Intellectually Rutherford is interesting as combining the study of philosophy with a taste for humane letters. He wrote on Aristotle, though he wrote—not in the Latin style of the Schoolmen, but in the style of the scholars of the Renaissance. During his residence in St. Andrews, Napier was boarded with Rutherford, who, from what we know of him, must have been an uncomfortable housemate. His temper was so violent that he kept his colleagues in perpetual hot water. The very year of Napier's stay in St. Andrews, Rutherford received a public rebuke from the authorities of the University for showing himself 'too hasty and impatient,' and he was admonished 'not to let the sun go down upon his wrath, and to study to bridle his tongue, and conduct himself with greater humility and mildness.' In spite of his unhappy temper, however, Rutherford was well fitted to impress the youth who sat at his feet, and doubtless Napier partly learned from him his interest in theology and philosophy as well as the direct and simple Latin style which he afterwards came to write.

In an interesting autobiographical passage in the address to 'The Godly and Christian Reader,' prefixed to his book on the Apocalypse, Napier tells us what was the dominant idea he carried with him from St. Andrews: 'In my tender yeers and barne age at Saint Androes at the Schools, having, on the one part, contracted a loving familiarity with a certain Gentleman, a Papist, and, on the other part, being attentive to the sermons of that worthy man of God, Master

Christopher Goodman, teaching upon the Apocalypse, I was so moved in admiration against the blindnesse of Papists, that could not most evidently see their seven-hilled city, Rome, painted out there so lively by Saint John as the mother of all spiritual whoredom, that not only burst I out in reasoning against my said familiar, but also from thenceforth I determined with myself (by the assistance of God's spirit) to employ my studie and diligence to search out the remanent mysteries of that holy book. . . .' The Christopher Goodman here mentioned was an Englishman, who had been a fellow-worker with John Knox in the Reformation first in Geneva and afterwards in Scotland, and was now settled as a Reformed minister in St. Andrews. He held the same opinions as Knox in religion and politics, was a Calvinist in doctrine, and believed in the right of subjects to rebel against their rulers under certain conditions. That Napier admired Goodman in youth and approved of him in maturer life throws a clear light on his own views regarding the burning questions that divided the men of his day.

It is uncertain how long Napier remained in St. Andrews, but, as he left the University without taking a degree, his stay there was probably short. There is some reason to believe that on leaving St. Andrews he studied for some years abroad, thus following a common custom of the time with the sons of Scottish nobles and gentlemen. Wherever he spent them, of these years we have no record; and the next mention we find of him is in 1571, when he was settled in Scotland. The intervening years had been portentous ones in the national history. They had seen Mary's marriage with Darnley, Darnley's murder, the dethronement of Mary, the coronation of her son James VI, and the establishment of a regency during his minority. In the years that were to come, till near the close of the century, one revolution in State and Church was to follow another, and Napier was to have his own share in both.

When we come upon Napier in 1571 he is domiciled, not in Merchiston Castle, but at Gartness, in the parish of Drymen in Stirlingshire, where his father possessed lands ; and here we are to imagine him as mainly residing till his father's death in 1608. In 1572 he married Elizabeth Stirling, daughter of Sir James Stirling of Keir, whose estate adjoined that of the Napiers in Menteith. About the same date we find Napier engaged in building a spacious mansion at Gartness, on the banks of the Endrick, with a garden, an orchard, and the usual offices attached to the mansions of country gentlemen of the time.[1] During the years 1570-2 raged the 'Douglas Wars' between the supporters of Mary on the one side, and the supporters of her son on the other. There is no more lamentable page in Scottish history. 'You should have seen,' writes the historian Spottiswoode, who was a child at the time, 'fathers against their sons, sons against their fathers, brother fighting against brother, nigh kinsmen and others allied together as enemies seeking one the destruction of another. . . . The very young ones scarce taught to speak had these words in their mouths and were sometimes observed to divide and have childish contests in that quarrel.' The Queen's party had gained possession of the Castle of Edinburgh, and its recovery was the main object of the party of the King. Edinburgh, therefore, was the centre of the internecine strife, and, as we have seen, Merchiston Castle was more than once roughly handled by both parties. Napier, however, we are to suppose, in his distant country retreat saw nothing of these wild doings, on the issue of which the future of the kingdom depended, for the capture of the Castle of Edinburgh by the King's party finally decided that Protestantism was to prevail.

[1] Some ruins still mark its site. According to the writer in the *New Statistical Account of Scotland,* it was a tradition in the neighbourhood that Napier was born at Edinbellie, in the parish of Balfron. A monument was even erected to commemorate the supposed fact.

But Napier was no absent-minded dreamer, indifferent to his own and to public affairs. From what we know of him, indeed, he appears to have been a man of counsel and even of action. In affairs of business he was the adviser of his own family, and he was consulted as an oracle by men in divers walks of life. As it happens, there are references in contemporary records to certain of his actions, which go to prove that in many respects he was a child of his age and well able to hold his own with its wildest spirits. These references bear both upon his public and his private action, and a few may be selected as throwing a curious light on the character of the man who is generally known to the world solely as the inventor of logarithms and a revealer of the mysteries of the Apocalypse.

We have seen how passionately Napier held the opinions of the Protestant party in politics and religion. For that party the policy of James VI was such as to keep them in chronic anxiety regarding its ultimate drift. James's one ambition was to succeed Elizabeth on the throne of England, whether by the aid of Protestants or Catholics was indifferent to him. His game, therefore, was to hold the balance, so far as he could, between his Protestant and Catholic subjects, the latter being still a powerful body in the kingdom. In 1593 a discovery was made which confirmed the worst fears of the ministers of the Reformed Church, and the action of Napier proved that he shared these fears to the full. The discovery made was that certain Catholic nobles and others were inviting Philip II of Spain to send an army to Scotland with the object of effecting the conquest of Britain—an enterprise which the great Armada of 1588 had so miserably failed to accomplish. The General Assembly of the Church, panic-stricken at the peril that now seemed impending, besieged the King with demands for immediate and effectual dealing with the enemies of Church and State. On three different occasions Napier accompanied deputations sent by the Assembly to lay its protest before the King. But he took a bolder step on his own responsibility—a step which proved not only the strength of his convictions but his courage in maintaining them. In the same exciting year, 1593, appeared his wonderful book, *A Plaine Discovery of the Whole Revelation of St. John*, and it was accompanied by a letter addressed to the King, the plain speaking of which was worthy of John Knox. One passage from it will sufficiently illustrate its tenor and its general tone. 'Therefore, Sir,' he wrote, 'let it be your Majesty's continuall study (as called and charged thereunto by God) to reforme the universall enormities of your country, and first (taking example of the princely prophet David) to begin at your Majesty's owne house, familie and court, and purge the same of all suspicion of Papists and Atheists and Newtrals, whereof this Revelation foretelleth that the number shall greatly increase in these latter daies. For shall any Prince be able to be one of the destroyers of that great seate, and a purger of the world from Antichristianisme, who purgeth not his owne countrie ? Shall he purge his whole countrie who purgeth not his owne house ? Or shall he purge his house, who is not purged himself by private meditations with his God ? ' Napier, it is clear, was a man of his time, with no hesitations regarding the absolute truth of his own convictions.

Other references in contemporary records show that with or against his own will Napier had to give his attention to other matters than mathematics and the Apocalypse. Thus we read, under the date 1591, that his father had to become surety for him that the Earl of Moray and others would sustain no harm from him. The reference throws light on the conditions under which Napier lived in his home at Gartness. The tenants of the different proprietors in the neighbourhood were at continual feud, seizing every opportunity of doing mischief to each other. Napier's tenants, it appears, had been specially aggressive, and a protest to the Privy Council had resulted in this demand on their landlord to look better after their future behaviour. That Napier was a stickler for his rights and resolute in upholding them is illustrated by several incidents in his life. In 1602 we find him lodging a protest on his own account before the Privy Council. The magistrates of Edinburgh, he complained, had illegally, though with the warrant of the Council, erected buildings for plague-stricken persons on his land in the district of the Sciennes. The complaint placed the Council in a dilemma. They had illegally granted a warrant for the erection of the buildings in question, but the presence of the plague made them necessary at the time—the Sciennes district then being at a considerable distance from the town. Necessity overriding law, the Council ordained that Napier should permit the magistrates to retain the use of the buildings till the following Candlemas, all his rights being conserved.

Napier was not so successful in another action which he brought before the Council. The date is 1613, when he had been Laird of Merchiston for five years, and the action is connected with his lands in the district of Menteith, which were managed by his bailie, by name John Mushat. The bailie's report on which the complaint was founded was to the following effect:—One Thomas Grahame had conceived 'a deadly hatred and malice against the Laird of Merchiston,' and had sought by all manner of indirect means to do him injury. Grahame was aware that the Laird of Merchiston was now 'heavily diseased with the pain of the gout and unable to repair to the said lands for holding of courts thereon, and doing of justice to his poor tenants who laboured the same,' and, availing himself of the opportunity, Grahame had grievously molested Merchiston's dependants. The charges against Grahame may be given, as they throw light on the world in which Napier lived. Grahame had evicted a tenant on Napier's lands on the ground that he had failed to pay his share of corn due to him; had 'after a very insolent manner' cut in pieces a plaid belonging to another tenant; had personally assaulted Napier's bailie while holding his regular court of justice; and finally had 'most barbarously gored' a horse belonging to a third tenant. The judgment of the Council was that the prosecution had 'succumbed and failed in proving of any point of the said complaint.'

Another litigation in which Napier was engaged illustrates the tenacity with which he maintained what he conceived to be his rights. His father, Sir Archibald, was Master of the Mint from 1576, and at his death in 1608 had in his possession certain documents and materials pertaining to his office. His successor in the Mint, one John Aitcheson, described as 'portioner of Inveresk,' demanded their delivery from John Napier, now Laird of Merchiston. The demand was refused, and Aitcheson brought the matter before the Privy Council. Sir Archibald on his deathbed, so Aitcheson alleged, had, in the presence of his son John, given orders that the documents in question should be delivered to Aitcheson as his successor in the Mint. Summoned to the bar of the Council, Napier produced twenty-six documents, but under the protest that he did so, not as heir to his father, but as having been delivered to him officially by the Sheriff-Depute of Edinburgh. Apparently, however, he did not on this occasion produce all the documents demanded, since an entry in the Register of the Council a year and a half later states that he eventually produced thirty-eight in all, when he received a full discharge for himself and his heirs.

The most singular affair in Napier's life that is known to us remains to be told, and it suggests curious reflections regarding the character of the man. As has been said, he passed among the people of his time for a dealer in the black arts, and many tales were told of his superhuman powers. He had as a familiar a jet-black cock, which had the uncanny gift of revealing to him the most secret thoughts of his domestics.[1] On one occasion he was reputed to have given triumphant proof of the creature's weird attributes. Some of his belongings had disappeared, and suspecting his domestics, he coated the cock with soot and shut it up in a dark room. Each domestic in turn was then ordered to enter the room and stroke the cock's back—having been previously informed that the cock would crow when the guilty person touched it. The cock remained silent throughout the procession, but the clean hands of one of the domestics conclusively proved that he was the criminal. Another story, current till recent times in the neighbourhood of Merchiston, shows the popular belief in his enchantments. The pigeons of a neighbouring laird provoked him by eating his corn; he protested, and threatened to poind (impound) them. 'Do so, if you can,' was the answer. The next morning the fields were covered with pigeons apparently under enchantment—their impounding by the magician's servants immediately following.

It was doubtless Napier's reputation for powers beyond those of ordinary man that in 1594 brought him into alliance with a dubious associate. This associate was Robert Logan of Restalrig, known in Scottish history by his connection with the famous Gowrie Conspiracy. As his past career had shown, Logan was a desperado, ready for any deed of devilry. He had obtained possession of Fast Castle, a wild fortress, perched on a precipitous rock overhanging the German Ocean, and of which James VI said that the man who built it must have been a knave at heart. In this stronghold Logan entertained associates of the same

[1] The Napiers were the hereditary Poulterers of the King, who was entitled to demand an annual gift of poultry from them. This fact may have originated the story of Napier's familiar.

feather as himself—highway robbery being one of their means of securing a livelihood. It is with this outlaw (for outlaw he was) that we find the author of *A Plaine Discovery* and the inventor of logarithms entering into a strange compact.

There was a tradition, common in the case of similar strongholds, that in Fast Castle a valuable treasure lay concealed. The finding of such a treasure would have been a godsend to Logan in his desperate fortunes, and as the one man in the kingdom likely to ensure a successful search, he applied to Napier. Napier responded, and with his own hand drew up a contract, which still exists. From the nature of the document, indeed, it could hardly have been entrusted to other hands. Here are the principal conditions agreed on by the two confederates. Napier was to do his utmost 'by the grace of God' to discover the treasure, and if it were found his share was to be a third. He was to be safely convoyed to and from Edinburgh, and on his return to Merchiston he was, in the presence of Logan, to destroy the contract. Should no treasure be found, Napier was to leave it to Logan to decide what satisfaction he should have for his pains. Such was the singular covenant to which we find the pious mystic of the *Plaine Discovery* pledging himself. The story does not suggest pleasant reflections, but we must remember the times when Napier lived. 'One century,' it has been said, 'may judge another century, but only his own century may judge the individual'; and this incident in Napier's life is a case for the application of the dictum. In point of fact, as the contract was not destroyed, we are to infer that it did not take effect. Be it added that subsequent documents from Napier's hand seem to imply that he afterwards carefully avoided all dealings with Logan and his kin.

A document of another kind, also from Napier's own hand, and still preserved in the library of Lambeth Palace, illustrates the restless ingenuity of his mind. It is a list of warlike engines which 'by the grace of God and worke of expert craftsmen' he hoped to produce 'for defence of this Iland.' These terrific engines, which were to render Britain safe from all her enemies, were as follows:—a burning mirror which would consume an enemy's ships 'at whatever appointed distance'; another mirror constructed on a different principle which would produce like effects; a piece of artillery which would sweep a whole field clear of an enemy[1]; a chariot which would be like 'a moving mouth of mettle and scatter destruction on all sides'; and finally 'devises of sayling under water, with divers other devises and stratagemes for harming of the enemyes.' But Napier's ingenuity was also turned to more innocent applications, and we are told that he was the first to suggest that salt was an efficacious fertiliser of the soil, and the first also to suggest certain novel methods of agriculture.

A few more contemporary references to Napier may conclude this sketch. In 1601 we find him becoming surety for a kinsman, resident in Edinburgh, that he will not have Mass said in his house. In 1608 he was appointed, along with another, to fix the price of boots and shoes within the burgh. In the same year, the year in which he succeeded his father, he was embroiled in a serious dispute with his step-brothers and sisters regarding the tithe sheaves of the lands of Merchiston. As the time approached for lifting the sheaves, both parties gave out that they meant to arm their respective

[1] Sir Thomas Urquhart of Cromarty thus describes this engine: 'He [Napier] had the skill to frame an engine . . . which by vertue of some secret springs, inward resorts, with other implements and materials fit for the purpose, inclosed within the bowels thereof, had the power . . . to clear a field of four miles circumference of all the living creatures exceeding a foot of height, that should be found thereon, how near soever they might be to one another, by which means he made it appear that he was able, with the help of this machine alone, to kill thirty thousand Turks without the hazard of one Christian. Of this it is said that (upon a wager) he gave proof upon a large plain in Scotland to the destruction of a great many herds of cattel and flocks of sheep, whereof some were distant from other half a mile on all sides and some a whole mile.'

friends and do open battle for their possession. Their intention came to the knowledge of the Privy Council, which appointed a neutral person to store the disputed sheaves in the barnyard of Wright's Houses, and forbade the contending parties, 'under pain of rebellion,' to touch them till the dispute was legally settled.

Napier died on April 4, 1617—apparently of gout, with which he had long been afflicted. His will, which was drawn up four days before his death, concludes with these words: 'With my hand at the pen led by the notaries underwritten, in respect I do not write for myself for my present infirmity and sickness.' The place of his burial is uncertain, but it was probably in the old church of the parish of St. Cuthbert's, Edinburgh.

Napier's mathematical works will be dealt with in other papers in this volume, but something may be said here of his book entitled *A Plaine Discovery of the Whole Revelation of St. John*, which, indeed, forms an essential part of his biography. In that book we come into closer touch with the man than in any other of his productions that are preserved. It is, in point of fact, a profession of his own faith regarding the temporal and eternal interests of man, set forth with all the power and fervour of which he was capable. We have seen how the train of thought which led up to it was originally prompted. The preaching of Christopher Goodman and his conversations with his Roman Catholic friend at St. Andrews determined him, he tells us, to devote himself henceforth to the study of 'that holy book,' the Revelation of St. John. After long and futile meditation on the mysteries of the book, an illuminative light dawned upon him, and he began to set down his interpretation in Latin. But the menace to Protestantism on the part of Rome, so alarmingly brought home by the Spanish Armada and kept alive by the temporising policy of

James, convinced him that the time had come when it was his public duty to bear his testimony. 'I was constrained of compassion,' he wrote in his address to the 'Godly and Christian Reader,' 'leaving the Latine, to haste out in English this present Work, almost unripe, that hereby the simple of this Island may be instructed, the godly confirmed, and the proud and foolish expectations of the wicked beaten down.'

The problem which Napier set himself to solve, and which, as we know, similarly exercised the genius of Newton, had a double attraction for him: it interested him as being naturally of a devout mind, and it interested him as a problem in the science of numbers. Both for him and for Newton the Bible was a verbally inspired volume given by God for man's salvation. Was it to be thought, therefore, that its interpretation was not permissible and possible to the mind of man? 'To what effect,' Napier asks, 'were the Prophecies of Daniel and of the Revelation given to the Church of God, and so many dates of years and circumstances of time foreshewing the latter day contained thereintill, if God had appointed the same never to be known or understood before that day come?'

The book is made up of two parts, the first consisting of thirty-six propositions, to each of which a proof is added, and the second of notes and commentaries on each verse in the Book of Revelation. The most momentous conclusion reached is that the Pope is Antichrist. The announcement of this fact is, indeed, declared to be the prime object of prophecy. The 'whole work of Revelation,' the author says, 'concerneth most the discovery of the Antichristian and Papisticall Kingdome.' Another conclusion he draws from his interpretation is the exact date of the end of the world. 'The last trumpet and vial,' he avers, 'beginneth anno Christi 1541 and should end anno Christi 1786.' 'Not,' he adds, 'that I mean that that age, or yet the world, shall continue so long, because it is said that, for the Elect's sake, the time shall be shortened; but, I mean that, if the world were to endure, that seventh age should continue until the year of Christ 1786.' He finally concludes 'that the day of God's judgment appears to fall betwixt the years of Christ 1688 and 1700.'

Such is the general character of the book which Napier certainly thought was of more importance to the world than his invention of logarithms. And the world at the time shared his opinion. Few books have commanded such general attention as the *Plaine Discovery.* Between 1593, the date of its publication, and 1645, five editions in English appeared; between 1600 and 1607 three in Dutch; between 1602 and 1607 nine in French; and between 1611 and 1627 four in German.[1]

From the foregoing sketch it will have appeared that materials do not exist for a connected narrative of Napier's life. In Napier's day it was not customary to give to the world a two-volume biography of every distinguished man immediately after his decease. He was fortunate, indeed, if some friend found occasion to write a brief outline of his career, such, for example, as we have in the case of George Buchanan. Of Napier we have not even such an outline from the hand of a contemporary. The incidental references to him which are preserved do not enable us to determine what manner of man he was in those closer relations with his fellows which reveal temperament and character. We have no clear glimpse of him as son or husband or father, as a friend or as an enemy. Great men, it has been said, should be regarded on their great sides; and the great, the salient traits of Napier's character are sufficiently in evidence from what knowledge we have of him. His most notable achievement—his invention of logarithms—has given him a high and

[1] These numbers are taken from W. Rae Macdonald's edition of *The Construction of the Wonderful Canon of Logarithms* (Edinburgh and London, 1889).

permanent position in the history of European culture, and to have attained such a position constitutes his indubitable claim to the remembrance of posterity. Genius is not always allied with an impressive personal character, but in Napier we find a breadth of humanity, a passionate interest in the welfare of his fellow-men which claim our regard apart from the special gifts that were allotted to him. No mere cloistral student, he threw himself with all his powers into the life of his time, and fearlessly, by word and deed, bore his testimony to the truth which he held to be of the highest concern for his countrymen and for the world at large. If we cannot now accept Hume's appreciation and regard Napier as the greatest man whom his country has produced, his name is at least one of the most distinguished in her annals, and in celebrating his memory she is only discharging a debt to one who is among her chiefest glories.[1]

[1] As a specimen of the dignified eloquence with which Napier wrote, we quote the concluding sentences of the address to the 'Godly and Christian Reader' prefixed to the *Plaine Discovery*.

'But forasmuch as this our good intention and godly purpose doth always proceed of a very tender and fraile vessell, and that, as all liquors (how precious soever) doth take some taste of their vessells, so this holy work may in some things (though not espied by myself) taste of my imperfections. Therefore humbly I submit these imperfections whatsoever to the gentle correction of every wise and discreet person, who in the motion of God's spirit judgeth uprightly, without envy or partiality, praying all good men to have me pardoned of whatsoever is amisse. For, although I have not done herein perfectly as I would, yet zealously as I could, knowing that the poor widow's mite was acceptable to the Lord ; for every man hath not gold, silver, silks, and purple to offer to the work of the Sanctuary ; to me (as saith Jerome) it should be much if I may purchase wool, or flocks to offer to that holy work. And, surely, this that I have, how small soever it be, till God enlarge me more, I offer it gladly unto the glory of God, and education of his true Church. To God, therefore, the disposer of this and all other godly works and meditations, who liveth and reigneth eternally in Trinity and Unity, be glory, praise, laud and thanks for ever and ever. Amen.'


A H

MERCHISTON CASTLE

George Smith, M.A., Head Master, Merchiston Castle School, from 1898 to 1914; now Master of Dulwich College

(Reprinted from the *Merchistonian*, 1912-13.)

So far as I know, no documentary evidence exists which would establish beyond a doubt the date of the building of the old tower. We must therefore fall back on indirect evidence and on a consideration of probabilities. First, a consideration of the style of architecture proves that it was certainly built *not later than* the sixteenth century. Secondly, although there is a considerable body of contemporary writings referring to the life of John Napier, the inventor of Logarithms, no mention occurs in any of these of the building of the tower: it is accepted as existing. As Napier was born in 1550, this brings us at least to the first half of the sixteenth century. But perhaps we can go even further back; for thirdly, we know from an old document still in existence that the estate of Merchiston had come into the possession of Alexander Napier before 1438. No mention is made of the Tower of Merchiston, but it would seem probable that it either existed already or was built soon after that date. I understand that the architectural evidence is not inconsistent with this early date: so perhaps we may believe that the old tower dates from the early half of the fifteenth century.

Mere numerical dates, however, may mean much or little. It is easier and more illuminating to remember that James I of Scotland had raised a loan of money from Alexander

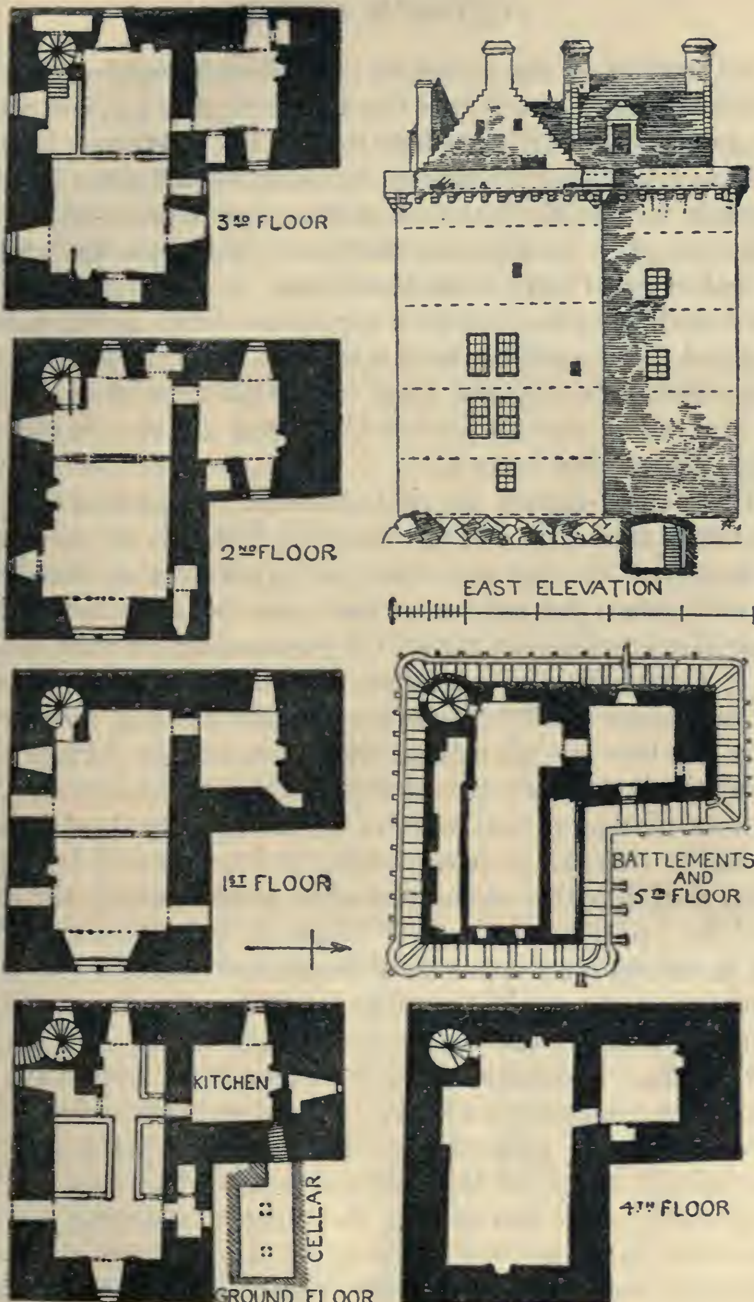
Napier, who was the Provost of Edinburgh in 1437, and had pledged the lands of Merchiston (originally part of the Crown demesne) in mortgage ('wadset' is the old Scottish term) for the repayment of the loan. As the loan was not repaid, the estate of Merchiston passed into the hands of the Napiers some time before 1438, the year after James I was murdered in the Blackfriars Monastery at Perth.

At that time the tower must have stood almost solitary.

'MARCHISTON TOWER'

'Published by S. Hooper, June 8, 1790. J. N. Sculp.'

Edinburgh consisted of the Castle, Holyrood House, and one long street—Lawnmarket, High Street, and Canongate—that ran down the ridge of the Castle rock to Holyrood House. The Cowgate—known as the Sou' Gate (South Gate)—was a country lane. About a mile to the south of the town lay the Borough Muir—bounded by the South Loch (now the Meadows) on the north, and by the King's Park on the east,



PLANS AND ELEVATION
From MacGibbon and Ross's *Castellated and Domestic Architecture of Scotland* by kind permission of Dr. Ross.

and extending on the south to the valley through which the Suburban railway now runs from Morningside to Newington. The moor lay on high ground, and at the western extremity of it stood Merchiston Tower, commanding the south-western approach to the city, as Craigmillar Castle commanded the south-eastern. In all directions save the east the ground slopes down and away from Merchiston, and an uninterrupted view could be had northward across the Firth to the hills of Fife, and southward to the Braids and the Pentlands. This view can still be enjoyed from the battlement of the tower, but to any one standing below that level the view is now cut off by intervening houses.

The ground-plan of the old tower is of the familiar L shape. The south side of it (facing Colinton Road) is 43 feet long; the west side (facing the west garden) is over 45 feet long; the east side (the only side that can be seen completely, facing the bowling green and the gymnasium) is fully 28 feet long. Originally the entrance to the tower was probably in the corner of the re-entrant angle looking north-east. There are two considerations which prove this. First, such a position of the main entrance made it easy to defend. The windows of that period were of course too small, at least in the lower stories, to permit of any foe entering by them, so the only possible entrance to the tower was by the door. Now the approach to a door situated within an angle like this is commanded on both sides by the defenders, and the attacking party could be subjected to a shower of missiles from the windows and loopholes both on the right hand and on the left. Secondly, there is a piece of architectural evidence that confirms this view. Round the battlement of the tower there are gargoyles or spouts—very plain and unadorned—to carry off the rain-water from the roof and the battlement. One can picture the old tower during a heavy rainstorm spouting out streams of water from miniature waterfalls, and any one walking within range of these would be drenched. But at the re-entrant angle we should see no water falling. The wall of the battlement is solid there. There are no gargoyles ; and the obvious reason for this was to save the inhabitants or friendly visitors from being drenched as they approached the door. We may therefore conclude that the main entrance was in that angle—probably where the passage is now between the Old Lower and the Boot Hall.

Some of my readers will remember the old plunge-bath of the School. It was built to the north of the Castle, about forty feet from the present wall of the class-rooms. It had always been a mystery where the water from it discharged when the bath was emptied. It was known that it did not discharge into the drainage system of the city ; and, although efforts had been made to trace it by putting colouring matter in the water and then examining possible outfalls, these efforts had been unsuccessful. In 1902, when the old baths were being demolished, the outlet pipe was traced. It was a very short one. About four or five feet from the old plunge-bath it discharged into a deep circular pit, the walls of which were built of solid old masonry and the bottom of which seemed to be strewn with rubble. In all probability this was the old well of the Castle. The spring which used to feed it had failed or been diverted ; and the well had apparently changed its function. Instead of supplying water to the Castle, it had for about forty-five years served as a pit into which the waste water of the bath was discharged, and from which it slowly oozed away into the ground.

Doubtless there were originally outworks and fortifications round the tower ; and we shall probably be justified in imagining that the space to the north at any rate was enclosed by a wall defending the water-supply of the Castle.

The old walls of the tower are not of uniform thickness. On most sides they are about six feet thick, but the short wall on the projecting north wing is about twice as thick, viz.

twelve feet. The reason for this is that the kitchen chimney was in this wall, and the kitchen chimneys of those days were spacious passages.

There is now only one face of the old tower which can be seen completely, namely the eastern, but even that does not give us a true view of what the original tower was like. The comparatively large windows on the first and the second floor are later alterations. The original windows were probably not much larger than that which can still be seen on the third floor.

The main staircase, part of which is still in use, is situated in the south-west angle of the building. It is a wheel staircase, partly built in the thickness of the wall and partly protruding from the rectangular corner as a segment of a circle. It ends in a turret, and a door opens outward from it on to the battlement.

There was at least one other staircase in existence. It seems to have run up the eastern side of the tower and was wholly contained in the thickness of the wall. It is very doubtful whether this staircase ran quite from top to bottom of the tower, but it is certain that it opened out into the middle room of the third flat. The opening used to be closed by an iron door; but in 1883, the jubilee year of the School, this door was removed and the recess was fitted up with drawers for storing clothes. The staircase can also be traced in the second flat. In the *Merchistonian*, vol. x, p. 67 (April 1882), it is said that this stair used to lead 'up from the dungeons below the Castle,' but I have not been able to find any traces of it in the building below the second flat, and it is most probable that it ended there. If there was a continuous stair from the dungeon (or cellar) to the third flat, it must have taken a very devious course through the old walls, piercing no fewer than three walls—the north-eastern, the north, and the eastern—in its progress; and this seems too elaborate.

There is one underground room. Those who wish to preserve the feudal atmosphere will call it a dungeon; the prosaic observer will probably call it a cellar. Curiously enough, it is not directly underneath the old tower, but under the square space at the re-entrant angle. The staircase which leads down to it opens out of the old kitchen. The floors and walls of the dungeon are cut out of the native rock. The roof is vaulted. It is the only vaulted roof in the Castle; the other roofs are supported on beams which rest on stone corbels projecting from the walls. There are two hatch-ways in the vaulted roof, and sometimes when the floor of the 'Old Lower' is not quite perfect, a few rays of light still trickle through. The door of the dungeon staircase is now hidden by a large cupboard, but once a year this cupboard is moved, and a voyage of inspection undertaken. So far as I know, no traces of human occupation by weary prisoners have been found—no staples for chains, no pathetic efforts at art, no doleful verses. It looks like a peaceful storage-cellar; but it may have been used for grimmer purposes.

Many changes have been made on the structure of the old tower since it was originally built. But these changes are all comparatively unimportant, and the tower shows practically all its original features.

Among the minor changes of structure which are most noticeable may be mentioned the east windows on the first flat and the second flat. These are comparatively large mullioned windows, and are, it is almost certain, not part of the original structure. This is proved in two ways. First, windows of such a size were not put into fifteenth-century houses; they would have weakened the defence. Secondly, along the line of these two windows there is a distinct vertical crack in the masonry, showing, I think, that a certain amount of slip or subsidence occurred when these windows were opened out; in all probability, the masons had not made sufficient allowance for the weight of the masonry which had now to be supported by beams.

The date at which these windows were put in was in all probability 1665, or a little later. It was in that year that Merchiston Castle passed out of the possession of the Napiers and came into the possession of the Lowis family. It seems probable that when the Lowis family entered on possession of it a good many alterations were made. It is certainly to this period that the plaster ceiling of the big room on the second flat belongs. This date is confirmed by the twice repeated C R 2 = Carolus Rex Secundus, or Charles the Second. The evidence of style of ornamentation also points to the conclusion that the panelling of this large room on the second flat and the panelling of the small room on the third flat was carried out at this same time. Possibly enough also the two pillars which now stand inside the small central gateway were carved and erected about this same period. These pillars originally stood on the north side of the Castle.

About the year 1750, when the Castle passed out of the possession of the Lowis family, we are to imagine it still standing without any buildings round about it. The front entrance is still towards the north, and the original L shape of the ground-plan is probably not yet obscured. But in the latter half of that century, the re-entrant angle formed by the L was filled up by a building which can be seen in the illustration on page 54. This building must have blocked the original front door, and possibly necessitated the transference of the main entrance to the south side of the tower.

Somewhere at the beginning of last century was built that addition on the south side of the tower which now forms the front of it. The structure is by no means unsubstantial, but the internal arrangements exhibited no ingenuity of architectural device. There was a front door and an entrance hall. On the right side as you entered there was a dining-room, on the left side there was a drawing-room, both of exactly the same size and entirely symmetrical. Above the dining-room there was a bedroom and dressing-room, above the drawing-room there was a bedroom and dressing-room, both of the same size and entirely symmetrical. In the external appearance of this building an attempt is made to keep it in harmony with the structure of the tower. But the attempt can hardly be said to be successful. The old tower was right-angled in plan, but this comparatively modern addition has semicircular ends. The old tower had a plain parapet, but the parapet round the roof of this is ornamented by machicolations. These must always have been obtrusively useless, and apparently they were found to give too great opportunities for the wind to play upon the roof; and consequently all the machicolations on the west side of this part have been built up. It was doubtless at this date that the two richly ornamented pillars which now stand inside the central gateway were transferred to their present position; and it was about this time also that the central gateway was itself re-erected at the place where it now stands.

Some time after the School was established here, in 1833, the re-entrant angle of the tower was filled with permanent buildings, namely the Old Lower, Sick Room, and the rooms above it. This was done in Dr Chalmers's time, probably about 1840; but I have been unable to discover the exact date. Nor have I been able to discover the date when the present broad staircase was built, from the first to the second flat, thus doing away with the use of the turret stair between these flats. I think that that was during Dr Rogerson's headmastership, that is, between 1863 and 1898; but I am not quite sure.

LOGARITHMS AND COMPUTATION

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Napier's name would be known in the history of scientific discovery even if he had not invented logarithms. His analogies and rules of circular parts would alone suffice to give him a place in mathematical history, and one which is of special interest to us, as they stand out as the earliest contributions to the exact sciences made in Great Britain. But his additions to spherical trigonometry are so completely overshadowed by his capital discovery or invention of logarithms that, especially on this occasion, it is natural to think only of the greatness of this achievement and its consequences.

By this invention Napier introduced a new function into mathematics, and in his manner of conceiving a logarithm he applied a new principle; but even these striking anticipations of the mathematics of the future seem almost insignificant by comparison with the invention itself, which was to influence so profoundly the whole method of calculation and confer immense benefits upon science and the world.

There is so much to admire in connection with Napier's grand achievement that I should be embarrassed to decide in which aspect it ought to appeal most strongly to the mathematician and calculator. Speaking for myself, however, nothing surprises me more than that Napier in the then state of algebra should have deliberately set before himself the task of simplifying the process of multiplication, and should have completely succeeded by discovering the only practicable mathematical method that exists.

In contrast to the *Canon Mirificus* consider for a moment the *Rabdologia.* Interesting as that work is, it is not an abnormal product of its time. During the previous half-century the dread of computation was so great that the tendency of English writers on arithmetic had been to replace computations by mechanical contrivances, rather than to perfect the rules for computing. The abacus had been in use from remote times, and we should expect that any one endeavouring to render multiplications less laborious would seek to do so by means of an apparatus. Napier's rods are merely a mechanical device to save the effort of multiplication, and at the present time their only use would be to avoid the necessity of remembering the multiplication table and applying it from memory.

But far different was the invention of logarithms. Quite apart from their power in other branches of arithmetic besides multiplication, there is a vast difference between devising an apparatus to perform or facilitate a definite operation and discovering a mathematical method by which it may be replaced by a simpler operation. The process of multiplication is so fundamental and direct that, from an arithmetical point of view, it might well be thought to be incapable of simplification or transformation into an easier process, so that there would seem to be no hope of help except from an apparatus. But Napier, not contented with such aids, discovered by a most remarkable and memorable effort of genius that such a transformation of multiplication was possible, and he not only showed how the necessary table could be calculated, but he actually constructed it himself. That Napier at a time when algebra scarcely existed should have done this is most wonderful; he gave us the principle, the method of calculation, and the finished table.

The greatest event in the history of arithmetic is the introduction into Europe of the Arabic numeration—the nine digits and cipher and the principle of local value; in fact, arithmetic in its modern sense begins with the Arabic notation. With such a system of numeration arithmetic not only gained the power of representing simply and uniformly all whole numbers, but it gradually acquired rules for performing operations. The most enduring work to which the new arithmetic was applied was the calculation of tables of sines and other trigonometrical lines. These laborious calculations were continued by various great mathematicians during the sixteenth century, and reached their climax in the magnificent tables of Rheticus, published in 1596 and (by Pitiscus) in 1613. In the following year Napier's *Canon Mirificus* appeared, and the whole course of mathematical calculation was suddenly changed. Some may have wished—perhaps even Napier himself—that the original table had been one of numbers and, as we should now say, to base 10; but, for myself, I cannot but feel pleasure that the course of events was exactly what it was, and that Napier's table was one of sines, connecting him with the great table-calculators of the previous century. His principal object was to facilitate the multiplication and division of sines, and this was effected by his table.

His logarithms, modified by the introduction of decimal points, would now be described as representing numbers to base e^{-1}. This base arose from his very remarkable and fundamental method of conceiving a logarithm, and gives a special interest to his table; but in the years during which he was occupied in devising methods of calculating logarithms and was actually computing the table, it is natural that he should have perceived the greater advantage of 10 as a base, and this we know to have been the fact from Briggs's account of his interview with him.

The history of the original calculation of decimal logarithms forms a worthy sequel to Napier's invention. I need not here do more than allude to Briggs's appreciation of the 'noble invention of logarithms lately discovered,' to his visits to Napier, to his vast logarithmic calculations begun soon after the publication of the *Canon Mirificus*, and continued without intermission till his death in 1631, or to the methods for the calculation of logarithms which he improved or devised.

There is every reason to believe that Napier calculated his canon with his own hand, and with it his work as a calculator was done; but in Briggs a most able and enthusiastic successor had arisen, and in default of Napier having been able to perform the calculations himself, nothing could have been more satisfactory than that the tables which were to render his invention universally available should have been constructed by one who had been his guest and had been encouraged by him personally in his great work.

Within twenty years of the publication of the *Canon Mirificus* the work of Briggs, supplemented by that of Vlacq, gave to the world the great fundamental logarithmic tables of numbers and trigonometrical lines which form the source from which, until quite recently, all subsequent tables have been derived.

It was a great thing to replace multiplications and divisions by additions and subtractions, but it was even a greater gain to computation to replace root-extraction by division. We may assume that Napier's original object was to simplify multiplications and divisions, but as soon as he had obtained the conception of a logarithm he would have seen that thereby the extraction of roots was reduced to division; and in the preface to the *Canon Mirificus* he explicitly mentions the extraction of square and cube roots as well as multiplications and divisions. Rightly indeed did Napier use the adjective *mirificus*.

Realising as he did the power of his invention, he was peculiarly diffident in putting it before the world, postponing the account of the construction of the *Canon* till he knew what reception it would meet with. Briggs's homage must have shown him how greatly his invention was appreciated, but I wish he could have lived to know of Kepler's enthusiasm.

More than fifty years ago, when a boy at school, I first saw a 7-figure table of logarithms, and was shown how to use it. My feeling of amazement at what it could do is a vivid memory to me. At that time I had learned arithmetic, and could extract square and cube roots, but had not reached logarithms in algebra; and it was a great wonder and mystery to me that the table before me could replace the extraction of these roots by division by 2 and 3. The mystery, of course, passed away, but the wonder has never left me, and although in all the years since then I have been interested in, and concerned with, mathematical tables and logarithmic computations, I have only admired more and more the table itself as a consummate piece of human effort, applying with singular effect mathematical principles to the general service of mankind.

The primary object of a mathematical table is to give results which can be taken directly from the table instead of having to be calculated when required. Thus an entire system of values is calculated once for all, individual numbers of which would otherwise have to be calculated *de novo* over and over again by each person who had need of them. The amount of labour originally bestowed upon the formation of a table is of no concern to the user, but the certainty of accuracy is of the highest importance to him; indeed, this certainty of accuracy in a tabular result is secondary only to the saving of labour afforded by the table.

When the revival of learning began in the fifteenth century, the attention of mathematicians was principally directed to the improvement of trigonometry (owing to the needs of astronomy), and much of the enterprise of the founders of modern mathematics was devoted to the calculation of tables of sines, tangents, etc. A table of sines is perhaps the best example of an important mathematical table. The tabular result is a quantity of frequent use, and the calculation of a single sine is difficult and laborious. Other examples are tables of square and cube roots, reciprocals, divisors of numbers, etc. In all tables of this class the tabular result is the object of the table, and is the quantity for which it is consulted. But in a table of decimal logarithms the tabular result is only a means to an end. The values of x for which 10^x is equal to the series of cardinal numbers are not of interest in themselves, but a table of such values of x is of inestimable power for its applications.

Thus tables of decimal logarithms belong to the class of subsidiary tables, but in this class they are supreme and unique. Subsidiary tables usually have reference to a particular problem, but logarithms have the very widest scope. They are available whenever quantities with integral or fractional exponents are connected by multiplication or division, and often afford the greatest help when the formula is least adapted for direct calculation.

Of the vast amount of abstract computation which has been carried out in the world since the introduction of the Arabic notation, it is safe to say that much the greater portion has been performed by the aid of logarithms. And not only have they so enormously facilitated calculation, but they have also been the cause of new researches, which otherwise would have been quite out of the range of what was practicable.

In science the main use of logarithms has probably been in astronomy and geodesy ; and only those who are actually concerned with gravitational astronomy, or are aware of the length and laborious nature of astronomical calculations, can appreciate the immense debt astronomy owes to logarithmic tables. But it is not only science that has benefited; the gain has been almost as great in the more practical subjects of navigation, life insurance, and engineering. Next to the astronomer, the principal user of logarithms has been the actuary.

The pure mathematician has had more concern with the logarithmic function than with the tables, and for him their principal use has been for the calculation of other tables of more strictly mathematical application.

Although certain methods presumably accompanied the Arabic notation on its introduction, the progress of arithmetic regarded as the simple art of computation was very slow, and it was a long time before organised rules for multiplication and division were established, the latter especially being regarded as a very difficult operation. The first English book on arithmetic preceded the *Canon Mirificus* by less than sixty years, and the difficulties of calculation in Napier's time may be inferred from his own *Rabdologia* and the welcome given to his rods, which remained in use for the next fifty years. The extremely limited algebra that existed differed little from arithmetic, and was almost without a notation. Napier uses + and −, which had been introduced in the middle of the sixteenth century, but no other signs. His results are expressed in words as proportions, and equations do not occur. It is a wonderful episode in mathematical history that logarithms should have been discovered at this early period, and it is even more surprising still that the invention should have taken the form that it did in Napier's hands.

I have no difficulty in perceiving that logarithms might have been introduced at that time in such a manner, as we know that Jobst Bürgi did actually conceive antilogarithms, *i.e.* as a correspondence between $(1{\cdot}0001)^r$ and $10r$;

for integral values of r, with interpolations; but Napier's conception of a logarithm was of a much more subtle kind, and involved the principle of a mathematical function. Even more remarkable, too, than the idea itself was his mathematical skill in giving effect to it by assigning limits to the value of a logarithm by means of which a table could be calculated.

Next to the invention itself perhaps the most striking fact in connection with it was the instantaneous appreciation it met with from mathematicians, especially in England. Neither England nor Scotland had up to this time taken any obvious part in the calculation of the natural trigonometrical canon on which the continental mathematicians had expended so much effort; but no sooner was the *Canon Mirificus* published than the latent computational power of Great Britain began to appear. Briggs was the first of a long series of zealous calculators who united mathematical ability of a high order with love of computation. More than ten years earlier he had completed a table of 15-place natural sines and of 10-place tangents and secants, which was still unpublished when his whole attention was diverted to logarithmic calculation. Gunter was the first to calculate decimal logarithms of sines and tangents; and, with the exception of Vlacq's work, the whole computation of the fundamental tables of decimal logarithms of numbers and trigonometrical lines was carried out in England.

The numerous methods of calculating logarithms form an interesting department in the history of mathematics. We know how Napier calculated his *Canon*, and the processes described in the *Constructio* have a unique interest as the actual means by which he passed from his conception to the table itself. Kepler used practically the same processes for the calculation of his table of 1624. The methods employed by Briggs were mainly those of Napier, but to him is due the so-called radix method, so often rediscovered since, which depends upon the resolution of a number into factors of the form $1+\frac{r}{10^n}$. He was also the first to formulate the method of differences and apply it systematically to the calculation of a table. His work has scarcely been adequately appreciated, for his fame as a calculator has unduly overshadowed his distinction as a mathematician.

No special improvement was made in Napier's or Briggs's methods for some time, but during the seventeenth century logarithmic calculation engaged the attention of Gregory, Nicolas Mercator, Newton, Halley, Brook Taylor, and Abraham Sharp, all of whom made valuable contributions. In the course of that century the connection of logarithms with the quadrature of the hyperbola was discovered, series were introduced, and logarithms began to be regarded as exponents instead of as measures of ratios.

This is not the place in which to give an account of the numerous calculations of logarithms which have been made; but I may refer to the few complete decimal tables which have been calculated *de novo* since the publication of the original tables of Briggs and Vlacq. The Tables du Cadastre (to fourteen places, twelve correct) were undertaken at the end of the eighteenth century. They have not been published, but in 1891 the French Government issued 8-figure tables, which were derived from them. These tables give logarithms of numbers to 120,000, and of sines and tangents for every ten centesimal seconds, the quadrant being divided centesimally. A 12-place table of logarithms of numbers to 120,000 was calculated by Mr John Thomson of Greenock, who died in 1855, but the table was not published. The late Dr Edward Sang of Edinburgh made very extensive calculations of logarithms, including a 28-place table of logarithms of primes to 10,037 and of some higher primes as well as of composite numbers to 20,000, and a 15-place table from 100,000 to 370,000. These remain in manuscript. His 7-place table of logarithms of numbers to 200,000, published in 1871, was mainly derived from his own calculations. All of these manuscript tables have been used to correct errors in Briggs and Vlacq. In 1891 M. J. de Mendizábel-Tamborrel published tables of logarithms of numbers to 125,000 (8 places) and of sines and tangents (7 or 8 places) for every millionth of the *circumference*, which were almost wholly derived from original 10-place calculations. A 10-figure table of logarithms of numbers to 100,000 was calculated by Mr W. W. Duffield and published in 1895-6.

In 1910 the great 8-figure tables of Bauschinger and Peters were published. They give logarithms of numbers to 200,000 and of the trigonometrical ratios to every second (sexagesimal). These tables were the result of an almost entirely new calculation, for which a special machine was constructed. Briggs's fundamental tables, though giving fourteen places, can be relied upon only to twelve. The logarithms of the first 20,000 numbers were not recalculated, but a completely new 12-place calculation was made of the logarithms of numbers from 20,000 to 200,000, thus overlapping Briggs's table from 90,000 to 100,000. The trigonometrical canon was derived from Briggs's table of 1633 by interpolation, some original 20-figure calculations being also made.

In the following year M. Andoyer published his magnificent 14-place table of logarithms of sines and tangents to every ten seconds (sexagesimal). This table was derived from a complete recalculation, made entirely by M. Andoyer himself, without any assistance, personal or mechanical.

The increased accuracy now attainable in astronomy and geodesy has brought into existence these fine tables. For nearly three centuries the 7-figure tables were sufficient, but they have ceased to meet astronomical requirements. Another example of the demand for logarithms to more places than seven is afforded by the photographic reproduction of Vega's *Thesaurus* of 1794 (ten figures), which was issued by the Geographical Institute of Florence in 1889.

Although I abstain from mentioning many striking logarithmic calculations or applications of logarithms (such as Gaussian logarithms), I cannot resist the desire to refer to Abraham Sharp's 61-place logarithms of the first hundred numbers and of primes from 100 to 1100, and to Wolfram's great 48-place table of hyperbolic logarithms (*i.e.* to base e) of primes and many other numbers up to 10,009.

It is difficult to form an adequate idea of the vast number of logarithmic tables that have been issued. At the head of the list is the *Canon Mirificus*. There are the great tables of Briggs, Vlacq, and Vega, to which must now be added those of Bauschinger and Peters and Andoyer. Then come the multitude of 7-figure tables, and the 7-figure collections of tables containing natural sines, etc., as well as logarithms. These 7-figure collections of tables form themselves into a regular main line of descent in almost every country. Then come a few 6-figure tables, the army of 5-figure tables often bearing the name of Vlacq, and the numerous 4-figure tables, small tables printed on a card, etc.

A complete bibliography of logarithmic tables would be a very large work. Some years ago I collected for a time data for such a bibliography, but though I had some knowledge of the extent of the subject I was continually being surprised by the number of fresh tables which came to light.

More than twenty years before the publication of the *Canon Mirificus* the method of prosthaphæresis had been discovered and was in use. This method consists in forming the product of two sines by a process equivalent to the use of the formula

$$\sin a \sin b = \tfrac{1}{2}\{\cos(a-b) - \cos(a+b)\},$$

and enables the multiplication of sines to be effected by means of addition and subtraction only. It may be said that this formula had its origin in this very purpose.

It is rather surprising that the method of quarter squares which depends upon the formula

$$ab = \tfrac{1}{4}(a+b)^2 - \tfrac{1}{4}(a-b)^2$$

should have escaped the notice of Napier and his contemporaries. By this method two numbers can be multiplied together with the aid of a table of squares, or better still, of quarter squares by means of addition and subtraction only. A table of quarter squares up to 200,000 enables five figures to be multiplied by five figures in this manner. As it happened, the first mention of this use of a table of squares appears to have been in 1690, and no table of quarter squares was published until 1817.

It is through the *Canon Mirificus* that the function log x first entered into mathematics. In connection with a primary mathematical function such as sin x or log x or e^x, it is always interesting not only to examine the manner in which it was actually introduced, but also to contemplate the different ways in which it might have made its first appearance if the course of discovery had been different. Thus the sine came into mathematics through geometry, but had it not done so it must have arisen in the development of analysis either as the sum of its expansion in powers of x, or in connection with e^x, or from the consideration of singly periodic functions, or from the solution of the differential equation

$$\frac{d^2u}{dx^2} + u = 0.$$

In the case of log x, if it had not been introduced at so early a stage in the history of mathematics, it would have necessarily entered into algebra, after the admission of fractional exponents, as the inverse of the exponential function. It would also have occurred in connection with the series $x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 -$ etc., which represents the area of a portion of a rectangular hyperbola, or as the value of the integral $\int \frac{dx}{x}$. This last definition is in effect equivalent to

Napier's; for his definition gives $\frac{dx}{dt}=-x\frac{dy}{dt}$ where y is the logarithm of x; and his methods of calculating logarithms amount to a practical determination of y for given values of x from the differential equation, $\frac{dy}{dx}=-\frac{1}{x}$.

At the present time log x as a mathematical function is perhaps most naturally defined almost in the very manner in which it arose from Napier's conception, viz. as $\int_1^x \frac{dt}{t}$, for $x>0$. It is a new transcendent, *i.e.* not expressible in finite terms by means of algebraical or circular functions. Unlike sin x, it is not a uniform function, but is many-valued; indeed, log x, even for real values of x, has the infinite number of values $\log x+2ni\pi$. It is the first many-valued function introduced into mathematics.

Nothing in the history of mathematics is to me so surprising or impressive as the power it has gained by its notation or language. No one could have imagined that such 'trumpery tricks of abbreviation' as writing + and − for 'added to' and 'diminished by,' or x^2, x^3, . . . for xx, xxx, . . . etc., could have led to the creation of a language so powerful that it has actually itself become an instrument of research which can point the way to future progress. Without suitable notation it would be impossible to express differential equations, or even to conceive of them if complicated, much less to deal with them; and even comparatively simple algebraical quantities could not be treated in combination. Mathematics as it has advanced has constructed its own language to meet its needs, and the ability of a mathematician in devising or extending a new calculus is displayed almost as much in finding the true means of representing his results as in the discovery of the results themselves.

When mathematical notation had reached a point where the product of n x'es was replaced by x^n, and the extension of the law $x^m x^n = x^{m+n}$ had suggested $x^{\frac{1}{2}} x^{\frac{1}{2}} = x$ so that $x^{\frac{1}{2}}$ could be taken to denote $\sqrt{x}$, then fractional exponents would follow as a matter of course, and the tabulation of x in the equation $10^x = y$ for integral values of y might naturally suggest itself as a means of performing multiplication by addition. But in Napier's time, when there was practically no notation, his discovery or invention was accomplished by mind alone, without any aid from symbols. We who live in an age when algebraical notation has been extensively developed can realise only by an effort how slow and difficult was any step in mathematics until its own language had begun to arise, and how great was the mental power shown in Napier's conception and its realisation.

Computation did not become an art until the various methods of performing operations were reduced to order, so that instead of each piece of calculation, however simple, being a special problem to be solved, it became a definite piece of work to be carried out by fixed rules. To the art of computation thus constituted decimal logarithms form a real addition. It is true that the power of a table of logarithms is limited by its number of figures and extent, *i.e.* a 7-figure table can only give results to 7-figures; but, subject to this limitation, logarithms bring a new principle to the aid of arithmetic, by which many of its operations are enormously facilitated.

In our days when the rules of computation are precise, and when the construction of instruments has reached a high state of efficiency, the processes of multiplication and other arithmetical operations can be performed by machines designed for the purpose. These apparatuses which save mental strain and time are effective aids to calculation, and they may be regarded as the modern successors to Napier's rods. They assist in the performance of an arithmetical process, but do not add to the processes of arithmetic.

It is uncertain when the Arabic notation was introduced into Europe, but since its introduction the art of numeration and calculation has received only two important additions: decimal fractions and logarithms. The latter is due to Napier, and the former to Stevinus; but Napier is not without some share in the improvement of decimal arithmetic. Stevinus, though he completed the system of Arabic numeration by extending it to fractional numbers, did not treat integers and fractions alike, for the digits of the decimals were not placed in simple juxtaposition so as to show their values solely by their places, but after each digit an exponent was placed to indicate its value, these exponents being of the same nature as the marks used to denote primes, seconds, etc., in the case of sexagesimal fractions. Any one who adopted Stevinus's decimals and worked with them could scarcely fail to observe that these cumbrous exponents might be omitted, and that all that was required was to denote by some mark the place where the units ended and the tenths began. This Napier did, and in the *Constructio* the decimals are separated from the integer by a point, as is now usual. He would therefore seem to be the first person who used, at all events in a printed book, a simple separator; and the separator he used was a point. Briggs, in the portion of the *Constructio* written by him, distinguishes the decimals by underlining them, and in manuscript he bent upwards the left-hand end of the subscript line to indicate with precision where the decimals began. Both methods are equally effective and almost equally convenient. For a time Briggs's method, with the upward stroke shown, obtained currency, but ultimately the use of the simple point became universal in this country, though the *Constructio* probably remained the solitary instance of its employment for a long time.

The substitution of a simple separator for the exponents may seem to have been a small matter and an almost obvious simplification, and so perhaps it appeared to Napier; but from an arithmetical point of view the improvement was one of importance, for Stevinus's exponents, besides being useless, spoilt the regularity of the system, which only became uniform when they were replaced by a simple separator. Stevinus made a great addition to arithmetic by introducing decimal fractions, and it may be that it did not occur to him to dispense with exponents, which were in use in sexagesimal fractions, or possibly he thought such an innovation too daring and that it would retard the adoption of the new fractions. Some such feeling may have been in the mind of Napier too, for in the *Rabdologia*, the last work published in his lifetime, he uses the sexagesimal exponents for the decimals in the statement of the results in the only three examples in which decimals occur. But in the *Constructio* the decimal point is precisely defined, and is frequently used, so that it is unlikely that he would have made any alteration had he lived to publish this work himself.

With respect to computation in general, it seems to me that every person who possesses some mathematical knowledge and is at all interested in calculation must feel acute pleasure in contemplating the existing system of numeration (by means of which all numbers, however great or however small, may be expressed with so few symbols) and the power of algebraical methods—especially infinite series—for affording numerical values of mathematical quantities to any degree of accuracy.

It is, I think, mainly due to exultation in the power of algebra to supply the formulæ, and of arithmetic to afford the means of calculating from them and expressing the result, that mathematicians have been tempted to calculate constants and logarithms to a great number of decimal places—many more than would ever be required in any application—and to calculate high coefficients.

No doubt the desire to obtain the values of these quantities to a great many figures is also partly due to the fact that most of them are interesting in themselves; for e, π, γ, log 2, and many other numerical quantities occupy a curious, and some of them almost a mysterious, place in mathematics, so that there is a natural tendency to do all that can be done towards their precise determination. Still, it seems to me that the attraction in such calculations consists mainly in the fact that it should be *possible* to obtain and express the results to such an astounding degree of accuracy.

I myself have yielded to the temptation of calculating quantities that interested me to a greater number of decimals than could ever be required, and can testify to the pleasure of admiring a long row of figures in a new result and realising what a veritable triumph of algebra and arithmetic they represent. I am afraid it would be almost impossible to frame a complete justification for such extended calculations, as we know that the figures will never circulate, and it cannot be seriously urged that it is of consequence to know which of the digits is the hundredth figure in π or log 2; but it can be pleaded that such calculations have exercised a fascination over great mathematicians, and notably over Gauss, nearly the greatest of all. I may also here mention Adams's calculations of the first 62 Bernoullian numbers, his 263-place value of Euler's constant, and his 273-place values of log 2, log 3, log 5, and log 7.

Logarithms have their field when ten or a less number of figures are required, and they are not directly available for calculations which extend to many places of decimals; but even in such work every calculator is glad when at length the diminishing terms come within the range of the tables; and throughout the whole calculation the verifications which logarithms afford by giving the first few figures, the place of the decimal point, etc., are always welcome.

The *Canon Mirificus* is the first British contribution to the mathematical sciences, and next to Newton's *Principia* it is the most important work in the history of the exact sciences that has been published in Great Britain, at all events until within the memory of living persons.

In whatever country the *Canon Mirificus* had been produced, it would have occupied the same commanding position, for it announced one of the greatest scientific discoveries ever made. But besides the place it owes to its intrinsic merit, it possesses an almost romantic interest as coming from almost the last quarter of Europe from which such a discovery could have been looked for at that time.

I am very glad that it should have been decided to celebrate the tercentenary of the publication of the *Canon Mirificus*, and that the celebration should be held in Edinburgh, where it was published, and near to Napier's home where logarithms were conceived, invented, and first calculated, and where Napier received Briggs's visits. Personally, as one who has yielded to none in a lifelong admiration of Napier and logarithmic tables, I feel it a great privilege to take part in this celebration, and as astronomy was the cause of the invention and has profited most by it, it is an additional satisfaction to me to be here as one of the representatives of the Royal Astronomical Society.

THE LAW OF EXPONENTS IN THE WORKS OF THE SIXTEENTH CENTURY

By David Eugene Smith, Ph.D., LL.D., Professor of Mathematics in Teachers College, Columbia University, New York, Librarian of the American Mathematical Society, Vice-President of the International Commission on the Teaching of Mathematics.

It is well known to all who have considered the work of Napier that he approached the subject of logarithms, or at least said that the subject might be approached, through a comparison of velocities of two moving points, rather than through a consideration of logarithms as exponents.[1] It is commonly said that the reason for the failure to recognise the exponential relation, at least explicitly, was due to the fact that the symbolism of algebra and arithmetic was not at that time sufficiently advanced to allow of the simple method of approach which is seen in our present teaching of the subject.

To a certain extent this is a correct impression, and yet it is a well-known fact that the four fundamental laws of operation with logarithms were frequently mentioned in connection with the parallel series

0	1	2	3	4	5	6	7
1	2	4	8	16	32	64	128

long before Napier's time, that the importance of these

[1] *E.g.* see Professor Gibson's paper on 'Napier and the Invention of Logarithms,' read before the Royal Philosophical Society of Glasgow, February 11, 1914 (see their *Proceedings*, vol. xlv, pp. 35-56), and reprinted in the *Handbook* to the Napier Tercentenary Exhibition; Professor Hobson's lecture on *John Napier and the Invention of Logarithms*, Cambridge, 1914; Lord Moulton's Inaugural Address, which opens this volume; and Dr Conrad Müller's monograph in *Die Naturwissenschaften*, 1914, Heft 28.

laws was asserted, and the possibilities of something beyond mere curious relations were suggested. Of course, to us the lower of these two series is merely

$$2^0 \quad 2^1 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2^5 \quad 2^6 \quad 2^7,$$

so that the upper row is simply a series of exponents, and in it we see without any difficulty that

$$2^3 \times 2^4 = 2^7,$$
$$2^7 \div 2^3 = 2^4,$$
$$(2^2)^3 = 2^6,$$

and

$$(2^4)^{\frac{1}{2}} = 2^2,$$

which are the fundamental laws of logarithms, and which are as true for any other base as for the base 2.

The object of this paper is not to call attention to the fact that such a relationship as

$$a^m a^n = a^{m+n}$$

had been frequently recognised from the time of Archimedes,[1] nor to state the well-known fact that the relation between the two series had long been known, but to bring together extracts from the most important works of various writers concerning the laws in which we can now see the whole theory of logarithmic computation.

It is always difficult to find just where any special idea starts in this world, and one will look in vain for the source of the one which relates the arithmetical and geometrical series in the way mentioned. It may readily have been suggested by a study of such series as appear in the works of Boethius,[2] and particularly in the *Mensa Pythagorica* which he gives and which was known to all the mediæval writers on arithmetic,[3] or it may have grown up gradually

[1] *Archimedis Opera Omnia*, 2nd Heiberg ed., vol. ii, p. 243; Heath, *The Works of Archimedes*, p. 230 (Kliem ed., p. 351).

[2] For example, in the *Arithmetica*, cap. xxv. See Venice ed., 1499, fol. 14, r; Friedlein ed., 1867, p. 113.

[3] Among the many discussions of this square array of the multiplication table, with its arithmetical progression at the top and its geometrical progression, often in red, down the principal diagonal, may be mentioned the one in Boncompagni's *Bullettino*, vol. xii, pp. 144-152.

from the study of series found in the works of many writers of the early Renaissance.[1]

But whatever the origin of the idea, it was well grounded early in the sixteenth century. Most writers refer to Stifel as the first to set forth the basal laws,[2] and we shall see that he did set them forth very clearly, but he was by no means the first to do so, nor did they first appear even in his century. Probably the best of the statements concerning them which appeared in the fifteenth century were those of Chuquet in *Le Triparty en la Science des Nombres*, written in 1484, and from which Estienne de la Roche copied so freely in his *Larismethique* of 1520. Chuquet expresses very clearly the relations

$$a^m a^n = a^{m+n}$$

and

$$(a^m)^n = a^{mn}$$

in connection with the double series to which reference[3] has

[1] For example, Clichtoveus, *Epitome* of Boethius, Paris, 1503 (ed. 1510, fol. 20); Pacioli, *Summa de Arithmetica*, etc., Venice, 1494, fol. 39; Grammateus, *Eyn new künstlich behend vnd gewiss Rechenbüchlin vff alle Kauffmanschafft*, Vienna, 1518, fol. G_1, v; Finaeus, *Arithmetica Practica*, Paris, 1530 (ed. 1544, fols. 17, 73); Feliciano, *Scala Grimaldelli*, Venice, 1526 (ed. 1629, p. 192); Willichius, *Arithmeticae libri tres*, Strassburg, 1540, p. 48; Lonicerus, *Arithmeticae brevis Introductio*, Frankfort, 1551; and the *Algebra* of Initius Algebras, '*ad Ylem geometram magistrum suum*,' an anonymous manuscript of the sixteenth century, published in the *Abhandlungen zur Geschichte der Mathematik*, Leipzig, 1902, Heft xii, pp. 479, 508, 544.

[2] Among them is Kästner, *Geschichte der Mathematik*, vol. i, p. 119, who has been generally followed in this matter. See also Th. Müller, *Der Esslinger Mathematiker Michael Stifel*, Prog., Esslinger, 1897, p. 16, where the author states: '"Dies ist das älteste Buch," sagt Strobel, "in welchem die Vergleichung des arithmetischen Reihe mit der geometrischen als der Grund der Logarithmen vorkommt."'

[3] 'Il convient poser plusȝs nõbres ꝑporcionalȝ cõmancans a l. constituez en ordonnance continuee cõme 1.2.4.8.16.32. &c. ou .1.3.9.27. &c. ¶ Maintenant convient scauoir que. l. represente et est ou lieu des nombres dõt le^r denõia^on est .0./2. represente et est ou lieu des premiers dont leur denomĩacion est .1./4. tient le lieu des secondȝ dont leur denomĩacion est .2. Et .8. est ou lieu des tiers .16. tient la place des quartz' (fol. 86, v, of the *Triparty*). This is taken from the copy made by A. Marre from the orginal manuscript. Boncompagni published it in the *Bullettino*, vol. xiii, p. 593 *seq.*, fol. 86, v, being on p. 740. Marre's copy is now in the library of the writer.

been made, calling special attention to the latter law as 'a secret' of proportional numbers.[1]

It is also difficult to say when a plan of this kind first appears in print, because it is usually hinted before it is stated definitely. Perhaps it is safe, however, to assign the honour to Rudolff. In his *Künstliche rechnung* of 1526 the double series is given and the multiplication principle is clearly set forth,[2] and inasmuch as this work had great influence on Stifel, who in turn influenced Jacob, Clavius, and Bürgi, we are justified in speaking of this as somewhat epoch-making.

The next writer to refer to the matter was probably Apianus,[3] who followed Rudolff so closely as to be entitled to little credit for what he did.[4]

Following Apianus, the first arithmetician of any standing who seems to have had a vision of the importance of this relation was Gemma Frisius,[5] whose *Arithmeticae Practicae Methodus* was by far the most popular textbook of the sixteenth century, about sixty editions of the work

[1] '¶ Semblement qui multiplie .4. qui est nombre second par .8. qui est nombre tiers montent .32. qui est nombre quiut . . . ¶ En ceste consideration est maĩfeste vng secret qui est es nombres pporcionalz. Cest que qui multiplie vng nombre pporcional en soy Il en viẽt le nombre du double de sa denominĩacion come qui ml'tiplie .8. qui est tiers en soy Il en vient .64. qui est six^e^. Et .16. qui est quart multiplie en soy. Il en doit veuir 256. qui est huyt^e^. Et qui multiplie .128. qui est le .7^e^. pporcional par .512. qui est le 9^e^. Il en doit venir 65536. qui est le 16^e^.'

[2] 'Nun merck wenn du zwo zalen mit einander multiplicirst / wiltu wissen die stat des quocients / addir die zalen der natürlichen ordnung so ob den zweyen mit einander gemultiplicirten zaleu gefunden / dz collect bericht dich. Als wen ich 8 multiplicir mit 16. muss komen 128. darumb das 3 vnd 4 so vber dem 8 vnnd 16 geschriben zusamen geaddirt 7 machen.' He gives several examples, but goes no further with the law.

[3] Peter Bienewitz, or Bennewitz, professor at Ingolstadt, and one of the few university professors of his time who gave instruction in arithmetic in the German language. His work, entitled *Eyn Newe Vnnd wolgegründte vnderweysung aller Kauffmansz Rechnung in dreyen bũchern,* appeared at Ingolstadt in 1527. The Pascal triangle first appears in print on the title-page of this book.

[4] See ed. 1537, fol. D_3, v.

[5] His name was Gemma Rainer, or Regnier, and he was born in East Friesland in 1508.

appearing between 1540 and 1601. Gemma gives the law with relation to the double array,

3	9	27	81	243	729
0	1	2	3	4	5,

saying that the product of two numbers occupies a place indicated by the sum of their places (3×9 occupies the $1+2=3$ place), and that the square of a number in the fifth place occupies the 2×5th place.[1]

The first arithmetician to take a long step in advance of Rudolff was the commentator on the latter's work, *Die Coss*.[2] It was not, however, in this work that the theory is set forth, but in the *Arithmetica Integra* of 1544.[3] In this book, notable for its scholarly treatment of various subjects, Stifel refers to the laws of exponents several times. At first he uses the series

0	1	2	3	4	5	6	7	8
1	2	4	8	16	32	64	128	256,

distinctly calling the upper numbers *exponents*, and saying that the exponents of the factors are added to produce the exponent of the product, and subtracted to produce the exponent of the quotient.[4] Further, he expressly lays down

[1] 'Si enim duos quoscunque ex his numeris inuicem multiplicaueris, productúmque per primum diviseris, producetur numerus eo loco ponendus, quē duo facta indicabunt . . .' (ed. 1553, fol. 17, r, and note by Peletarius, fol. 78, v). The relation is not so clear as in some of the other texts on account of the arrangement of the series.

[2] *Die Coss Christoffs Rudolffs*, Königsberg, 1553. The original work was published by Rudolff at Strassburg in 1525.

[3] *Arithmetica Integra. Authore Michaele Stifelio*, Nürnberg, 1544. He also published a *Deutsche Arithmetica* at Nürnberg in 1545, in which he touches slightly upon the subject (fol. 61).

[4] 'Qualiacunqʒ facit Arithmetica progressio additione, & subtractione, talis facit progressio Geometrica multiplicatione, & diuisione. ut plene ostendi lib. 1. capite de geomet. progres. Vide ergo,

0.	1.	2.	3.	4.	5.	6.	7.	8.
1.	2.	4.	8.	16.	32.	64.	128.	256.

Sicut ex additione (in superiore ordine) 3 ad 5 fiunt 8, sic (in inferiore ordine) ex multiplicatione 8 in 32 fiunt 256. Est autem 3 exponens ipsius octonarij, & 5 est exponens 32 & 8 est exponens numeri 256. Item sicut in ordine superiori, ex four laws, namely, that addition in arithmetical progression corresponds to multiplication in geometrical progression, that subtraction corresponds to division, multiplication to the finding of powers, and division to the extracting of roots.[1]

And finally, Stifel not only set forth the laws for positive exponents, but also saw the importance of the negative exponents, using the series

−3	−2	−1	0	1	2	3	4	5	6
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64

and making the significant remark: 'I might write a whole book concerning the marvellous things relating to numbers, but I must refrain and leave these things with eyes closed.'[2] What had Stifel in mind when he wrote these words? What did this seer of visions in the world of religious mysticism see in this world of number mysticism? With both positive and negative exponents before him, with the four laws distinctly stated, with a recognition of other bases than 2—when he says that he recognises much more in all this, what was the region which he dimly saw beyond?

In speaking of the German writers of the middle of the sixteenth century, the fact should be mentioned that the

subtractione 3 de 7, remanent 4, ita in inferiori ordine ex diuisione 128 per 8, fiunt 16' (fols. 236, 237).

It will be noticed that he speaks of 8 as 'exponens numeri 256,' and not as the exponent of 2, but this has no significance with respect to the theory.

[1] '1. Additio in Arithmeticis progressionibus respōdet multiplicationi in Geometricis. Vt, sicut in hac Arithmetica progressione, 3.7.11.15. duo termini extremi additi, faciunt quantum medij ad se additi, utrobiq; enim fiunt 18. Sic in hac Geometrica, 3.6.12.24. duo extremi inter se multiplicati, faciunt quantum medij inter se multiplicati, utrobiq; enim fiunt 72. & sic de infinitis alijs exemplis.

'2. Subtractio in Arithmeticis, respondet in Geometricis diuisioni . . .

'3. Multiplicatio simplex (id est, numeri in numerum) quae fit in Arithmeticis, respondet multiplicationi in se, quae fit in geometricis. Vt, duplatio in Arithmeticis respondet in Geometricis multiplicationi quadratae . . .

'4. Diuisio in Arithmeticis progressionibus, respondet extractionibus radicum in progressionibus Geometricis' (fol. 35).

[2] 'Posset hic fere nouus liber integer scribi de mirabilibus numerorum, sed oportet ut me hic subducā, & clausis oculis abeā.'

famous horse-shoeing problem occasionally led to a recognition of the law of exponents. For example, Adam Riese states this law quite clearly in the fourth of his arithmetics, the one which appeared in 1550. In his solution of this problem he gives the usual arrangement of powers of 2, with the indices in a separate column, as found in numerous manuscripts of the fifteenth century,[1] and calls attention to the fact that he can find the number corresponding to 17 by multiplying the numbers corresponding to 9 and 8.[2]

A number of French writers of this period were also aware of the law, and Peletier[3] (1549) stated it clearly for the case of multiplication. Five years later Claude de Boissière elaborated this treatment, and spoke of the 'marvellous operations' which can be performed by means of the related series.[4] Two years after Boissière's work was

[1] For example, see Smith, *Rara Arithmetica*, p. 447. For a description of Riese's work see p. 250.

[2] 'Ich setz die Summa des fünfften nagels ist 16 die multiplicir in sich homen 256 duplir 4 die neben den 5 stehn wird 8 also hastu in den 256 den neunden nagel / dan 5 vnd 4 macht 9 multiplicirstu den neunden nagel in sich / als 256 entspringt der 17 nagel / dan 9 vnd 8 machen 17 wird 65536 / So du diesen in sich multiplicirst hastu den 33 nagel / dan 17 vnd 16 machen 33 wird 4294967296' (*Rechenung nach der lenge / auff den Linihen vnd Feder*, fols. 53, 54).

Among the writers of the same period one other may possibly deserve mere mention, Lonicerus, *Arithmeticae brevis Introductio*, Frankfort, 1551.

[3] The first edition of *L'Arithmetiqve* appeared in 1549. The extract here given is from the 1607 edition, p. 67. In speaking of the series

3	6	12	24	48	96
0	1	2	3	4	5

he says: 'Ie sçauoir qui est le nõbre qui eschet au neufieme lieu en ceste Progression Double Ie diuise 48, qui est sur 4, par le premier nombre de la Progression, 3: prouiennent 16: lesquels je multiplie par 96, qui est sur 5: (car 4 & 5 font 9) prouiendront 1536, qui sera le nombre à mettre au neufieme lieu.'

[4] He first gives the series in this form:—

3.	6.	12.	24.	48.			
0.	1.	2.	3.	4.	5.	6.	7.

He then says: 'Par ces nombres pourras faire merueilleuses operations: car si tu prens deux d'iceux quelconque qu'il soit, & multiplies l'vn per l'autre, prouiẽdra vn nombre lequel si tu diuises par le premier de la progressiõ, rapportera le nombre conuenant en la loge laquelle te mõsteront les deux nombres escrits dessous. Exemple. Ie presuppose ne sçauoir point le nombre qui est au dessus de 4, à sçauoir published the theory was again given by Forcadel (1565), with a statement that the idea was due to Archimedes, that it was to be found in Euclid, and that Gemma Frisius had written upon it.[1] Ramus, combination of scholar and dilettante that he was, recognised its value but added nothing to it or to its possible applications.[2] When, however, Schonerus came to write his commentary on the work of Ramus, in 1586, a decided advance was made, for not only did he give the usual series for positive exponents, but, like Stifel, he used the geometrical progressions with fractions as well, although not with the negative sign in the former series. Further, he used the word 'index' where Stifel had used 'exponent,'[3] and, like this great writer, gave

48: ie le veux trouuer par le moyen des precedens: voyant donc que le nombre requis est pour mettre sur les deux quantitez, lesquelles adioustées pourront faire 4, leur respondent, à sçauoir 6 & 24: & multipliant l'vn par l'autre, prouiendront 144: lesquels diuisez par le premier de la progression qui est 3, rapporteront 48, qui est le nombre requis en la quatriesme loge, specifée par les deux nombres dessous, qui sont 1 & 3, lesquels ioincts valent 4' (*L'Art D'Arithmetiqve,* Paris, 1554; ed. 1563, fol. 67).

[1] 'Il nous faut maintenant escrire commẽt l'on pourra trouuer le terme ou le nombre de tel lieu qu'õ voudra d'vne progression geometrique, apres auoir connu le premier terme ou commencement d'icelle, & la quantité de la raison continuée, comme s'ensuict.' He then gives the method, and says: 'Ce qui est demonstré d'Archimede au liure du nombre de l'arene, & de nous en interpretant l'arithmetique de Gemme Phrison, & au liures de l'Arithmetique d'Euclide' (*Arithmetiqve entiere et abregee,* Paris, 1565, p. 152).

[2] In his *Arithmeticae libri dvo,* Basel, 1569, he says of the series

	1.	2.	3.	4.	5.	6.
1	2	4	8	16	32	64

'Itaq; si quæras terminum progressionis octavum, adde arithmeticos terminos constituentes septimum, ut 3 & 4, & multiplica geometricos iis respondentes 8 & 16. facies 128 octavum terminum progressionis. Sic erit in hac progressione tripla

	1	2	3	4	5
1	3	9	27	81	243

Si quaeras decimum, multiplica 243 per 81' (1st ed., 1569, p. 46; ed. 1580, p. 50). There was an English translation by William Kempe in 1592.

[3] He first gives the series

	I.	II.	III.	IIII.	V.
1.	2.	4.	8.	16.	32.,

and then says: 'Vides duobus datis 8. & 16 respondentis indices 3 & 4 facere 7 evidence of an appreciation of the importance of the law.[1] In general the French writers already named, and in the list should also be included the name of Chauvet,[2] paid no attention to any of the laws except that of multiplication, while the German writers, following the lead of Stifel, took the broader view of the theory. This was not always the case, for Suevas, writing as late as 1593, did not go beyond the limits set by most of the French arithmeticians,[3] but in general the German writers were in the lead. This is particularly true of Simon Jacob, who followed Stifel closely, recognising all four laws, and, as is well known, influencing Jost Bürgi, whose *Progress Tabulen* was not published until

Dices igitur 128 factum multitudinis termino tertio per quartum esse multitudinis terminorum septimum' (ed. 1586, p. 199). He then gives the series

	I.	II.	III.	IIII.	V.
1.	$\frac{1}{2}$.	$\frac{1}{4}$.	$\frac{1}{8}$.	$\frac{1}{16}$.	$\frac{1}{32}$.,

and says: 'Si quaeras tertium minutum $\frac{1}{8}$ per quartū $\frac{1}{16}$, quodnam faciat minutum & cujus loci? Respondebo facere $\frac{1}{128}$ quod indicum totus arguit esse septimum minutum progressionis universae terminū octavum' (*ibid.*).

[1] 'Hic inveniendi termini modus fundamentum est omnis figuratorum proportionalium per Algebram & Logisticam numerationis' (*ibid.*).

[2] *Les Institvtions de l'Arithmetiqve*, by Jacques Chauvet Champenois, Paris, 1578. He gives the law (p. 68) as follows:—

'1.	3.	9.	27.	81.	243.	729.	2187.	6561.	19683.
0.	1.	2.	3.	4.	5.	6.	7.	8.	9.

Pour trouuer vn nombre progressif en certain ordre plus hault, sans se trauailler à trouuer ceux du milieu. Comme si lon vouloit trouuer le nombre qui eschet au 15. lieu, fault prendre deux des figures inferieures, lesquelles ioinctes ensemble facent 15. Comme 7. & 8. font 15. & 6. & 9. font le mesme 15. Et si on multiplie le nombre qui est dessus 7. qui est 2187. par celuy qui est dessus 8. qui est 6561. le produict donnera 14348907. pour le 15. lieu de la Progression. Les mesme fust aduenu, si on eust multiplié les nombres qui sont dessus 6. & 9. sçauoir 729 par 19683.'

[3] In his *Arithmetica* of 1593 he gives the series

0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.
1.	2.	4.	8.	16.	32.	64.	128.						8192.,

and says: 'Nun für dich die bey den Zahlen / unter der 6. vnd 7. stedt / Nemlich: 64. vnd 128. Dieselbigen vermehre mit einander / so kommen 8192. die setzt vnter die 13. stedte / dann 6. vnd 7. macht dreytzehen.'

1620, although conceived a number of years earlier.[1] This influence is explicitly acknowledged in the unpublished introduction to the work, which is given in manuscript in the copy belonging to the Danzig library, and lent to the Royal Society of Edinburgh for exhibition at this Celebration. We are, therefore, justified in saying that Bürgi was led to his discovery through the works mentioned above, while Napier approached logarithms through considerations seemingly quite geometrical or mechanical. But it was Clavius (Klau) of Bamberg, the learned German Jesuit, then living in Rome, who brought the theory to the highest degree of perfection reached in the sixteenth century. His first discussion of the subject appeared in the *Epitome Arithmeticae Practicae*,[2] published in 1583, and to this he was probably led by his essay on the *Arenarius* of Archimedes, which appeared in 1570.[3] In this he considers only multiplication and involution.[4] Much more than this was, however, in his mind, for in the *Algebra* which he published in 1608 he gave the series of positive and negative integral

[1] Jacob published *Ein New vnd Wolgegründt Rechenbuch auff / den Linien vñ Ziffern* at Frankfort in 1560. The editions examined are those of 1565, fol. 15, and 1600, fol. 15. He had already published another arithmetic in 1557. His statement concerning the exponential relation is as follows :—'So merck nun / was in Geometrica progressione ist Multipliciern, das ist in Arithmetica progressione Addiern / vnd das dort ist diuidiern das ist hie Subtrahiern / vnd was dort mit sich ist Multipliciern / ist hie schlecht Multiplicieren / Letztlich was dort ist Radicem extrahirn / das ist hie schlechts Diuidiern mit der zahl die der Radix in Ordnung zeigt.'

[2] Rome, 1583; ed. 1585, p. 291; Italian ed. 1618, p. 246.

[3] It was included in his edition of Sacrobosco's *Sphaera*. The essay on the *Arenarius* was translated into English by Anderson in 1784.

[4] He gives the series

1.	2.	4.	8.	16.	32.	64.	128.	256.	512.	1024.	2048.
0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.,

and says : 'Si numerus 32. in se multiplicatur, procreabitur numerus 1024. supra 10. collocandus, quod numerus 10. duplus sit numeri 5. quo numero 32. subscribitur. Item ex multiplicatione 8. in 256. producetur numerus 2048. supra 11. reponendus, quod numerus 11. componatur ex. 3. & 8. qui numeri sub 8. & 256. scripti sunt.'

powers of 2, thus including fractions as Stifel had done,[1] and showed a complete knowledge of the theory. He afterwards (p. 74) related the whole subject to the theory of positive exponents, but, of course, without the exponential notation.

It may be remarked in passing that one cannot examine the algebras and higher arithmetics of the sixteenth century without wondering that men like Tonstall,[2] Tartaglia,[3] and Cardan [4] could have made so little of the relationship which others so clearly recognised. They represented, however, the Latin school, and based their work almost wholly upon Archimedes, while the Teutonic theory, starting with Rudolff, and advancing through the works of Stifel, Jacob, and Clavius, approached so near to the invention of logarithms based upon exponents that it is a cause for wonder that they did not anticipate the works of Napier and Briggs. That they did this cannot be claimed for a moment, except as they led to Bürgi's discovery, but that their work deserves recognition with respect to the *Progress Tabulen*, and on an occasion of this kind, must be quite evident.

[1] He gives the following series:—

Etc.	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	Etc.
Etc.	$\frac{1}{128}$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	Etc.

He then writes in much the same style as Stifel; for example: 'Vt enim ex multiplicatione $\frac{1}{4}$ in 32, fiunt 8, ita ex additione − 2, ad 5. fiunt 3' (p. 29).

[2] See his *De Arte Supputandi*, London, 1522; ed. 1529, fols. 193, 194.

[3] See *La Seconda Parte del General Trattato*, which appeared in 1556. Several times in this work he shows that he is perfectly aware of the relation of the two series, but he has no such explicit statement as those given by Stifel and Clavius.

[4] See his *Practica Arithmeticae* of 1539, fols. G_8, k_1, caput 42, 'de proprietatibus numerorum mirificis,' and caput 43, 'de misticis numerorum proprietatibus.'


A H

ALGEBRA IN NAPIER'S DAY AND ALLEGED PRIOR INVENTIONS OF LOGARITHMS

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In discussing the invention of logarithms before this learned assemblage I feel that I am only carrying coals to Newcastle or, more appropriately expressed, I feel I am only bringing owls to Athens.

Few inventors have a clearer title to priority than has Napier to the invention of logarithms. In the history of science it is the rule, rather than the exception, for two or more men independently to develop the same ideas. For instance, the law of the conservation of energy was announced about the same time by scientists of different nationalities. Says Arago: 'I doubt whether it were possible to cite a single scientific discovery of any importance which has not excited discussions of priority.'[1] Nor are such coincidences at all strange. Scientific thought on a particular topic advances to a point where a certain discovery or invention may logically follow. Small wonder, then, if the new territory beyond is entered about the same time by more than one man of genius.

In the sixteenth century the exponential notation of modern algebra was not yet in vogue. But geometrical and arithmetical progressions were studied in their relation to each other, particularly by the German algebraist Stifel in his *Arithmetica Integra*, Nürnberg, 1544. Moreover,

[1] F. Arago, *Biographies of Distinguished Scientific Men*, First Series, Boston, 1859, p. 383.

about a century before the invention of logarithms the Portuguese Alvarus Thomas,[1] in a work entitled *Liber de Triplici Motu*, and published in 1509, divides the segment of a line into parts representing the terms of a convergent geometric progression; that is, a segment AB is divided into parts such that $AB : P_1B = P_1B : P_2B = \ldots = P_iB : P_{i+1}B = \ldots$ Napier does not mention either Stifel or Thomas. He probably had not seen their works; the ideas to which we have made reference were without doubt common knowledge among mathematicians of the time. At any rate, the arithmetical theory of the two progressions, together with the graphic representation of their terms upon straight lines, are foundations upon which Napier built his theory of logarithms. In addition, Napier used the mechanical concept of the velocity of flowing points. The Swiss mechanician and mathematician, Joost Bürgi, likewise used the two progressions in constructing his *Progress Tabulen* of 1620.

Besides Napier and Bürgi two others have been named by certain writers as candidates for the honour of having invented logarithms. One is the Danish astronomer, Christian Longomontanus, the other is the noted English mathematician, Edward Wright. The latter has not been known to modern historians as a possible rival inventor of logarithms. The claims made on his behalf were urged by an eighteenth-century writer, Benjamin Martin of London. It is worthy of note that neither Edward Wright nor Longomontanus himself ever laid claim to the invention of logarithms.

I hold in my hand a booklet published by Benjamin Martin in 1772, entitled *System of Logarithms*,[2] in which

[1] H. Wieleitner, 'Zur Geschichte der unendlichen Reihen im christlichen Mittelalter' in *Bibliotheca mathematica*, 1914, Bd. 14, Folge 3, p. 153.

[2] Benjamin Martin, *The Mariner's Mirror, Part III. Being a New and Compendious System of Logarithms, in all the different Kinds, viz., I. Nautical Logarithms invented by Mr. Wright, II. Natural Logarithms by Lord Neper, III. Common Logarithms by*

'proofs' are given to show that Wright published a table of logarithms before Napier. Benjamin Martin was a voluminous compiler of books on scientific subjects. In a catalogue of second-hand books I have counted thirty-one different publications attributed to him. Augustus De Morgan estimates Martin in these words [1]: 'Old Ben Martin (as his admirers called him) was an able, and in this instance [2] a concise writer. He wrote on every mathematical subject (and never otherwise than well, I believe, except on biography), and a complete set of his works is rarely seen. He was a bookseller.' Martin is pleasantly referred to in the eighth edition of the *Encyclopædia Britannica*.[3] Doubtless with excessive generosity in praise, Charles Hutton states that Martin 'became one of the most celebrated mathematicians and opticians of his time.' [4]

Now what has Benjamin Martin to say on the invention of logarithms? In the preface to his *System of Logarithms*, referred to above, he says:

'The Invention of Logarithms has been unjustly ascribed to Lord Neper, a Scotch Baron, whereas it is in Reality due only to our Countryman Mr Edward Wright, as I have fully demonstrated. Mr Wright's Table of Latitudes was the first System of Logarithms the World ever saw, and are peculiarly adapted to the Service of Navigation; whereas the Logarithms afterwards published by Neper, were of such a Nature, and Form, as rendered them of little Use, till they received a Transformation by

Mr. Briggs. With Their Application in the Operations of Arithmetic, the Doctrine of Ratios, Natural Philosophy, the Cotesian Geometry; and Navigation, in Particular. The Whole illustrated by the Logistic Curve at large; with the Construction and Delineation of all the Logarithmic Lines and Scales. London, 1772.

[1] Augustus De Morgan, *Arithmetical Books*, London, 1847, p. 73.

[2] De Morgan is here referring to Benjamin Martin's *New and Comprehensive System of Mathematical Institutions*, London, 1759.

[3] *Encyclopædia Britannica*, 8th ed., art. 'Martin, Benjamin.'

[4] Hutton's *Mathematical Dictionary*, vol. ii, London, 1795, art. 'Martin, Benjamin.'

Mr Henry Briggs, who might just as well have made the Trigonometrical Canon of Logarithms in present Use from Mr Wright's, as from Lord Neper's Form and Module, if he had but considered it as thoroughly as the perspicacious Dr. Halley did.'

In another place [1] Martin states that, besides the natural logarithms and the Briggian logarithms, there is a 'third System of Logarithms invented and published by our Countryman Mr Ed. Wright, before the Year 1610; and is therefore the *first* System of Logarithms that was ever invented.' This third system he calls the *nautical* logarithms. Martin proceeds to explain how Wright came to construct those logarithmic figures. In Mercator's chart the meridians of longitude are all straight lines parallel to each other; hence the parallels of latitude become larger than they should be. In order to preserve the proper position and bearing of a place on Mercator's chart, it is necessary that each small arc of the meridian be correspondingly enlarged. This may be done by multiplying the length of the small arc by the secant of the latitude. The meridian consisting of such enlarged arcs was called the 'nautical meridian.' Edward Wright in his *Certaine Errors in Navigation corrected*,[2] 1599, published a Table of Latitudes, giving numbers expressing the length of an arc of the nautical meridian. Wright computed this table by the continued addition of the secants of 1″, 2″, 3″, etc. Thereby he secured an approximation sufficiently exact for the mariner's use. Martin proves in his book that Wright's Table of Latitudes may be interpreted as being a table of the logarithms of the tangents of half the complements of latitude. In modern symbols the proof runs thus:

[1] Benjamin Martin, *op. cit.*, p. 12.

[2] *Certaine Errors in Navigation, Arising either of the ordinarie erroneous making or vsing of the sea Chart, Compasse, Crosse staffe, and Tables of declination of the Sunne, and fixed Starres detected and corrected.* By E. W., London, 1599.

A second edition, enlarged, appeared in 1610.

$r\int_0^\theta \sec\theta\, d\theta = r \log\tan\frac{90^\circ-\theta}{2}$. The modulus of Wright's alleged system of logarithms is ·3438; its base[1] is 18·33.

Before passing judgment on the claim made by Martin that Wright should be given priority over Napier in the invention of logarithms, we shall outline more fully the history of Wright's Table of Latitudes. According to Edmund Halley[2] it was Henry Bond who discovered by chance that Wright's table was analogous to a scale of

[1] The nature and arrangement of Wright's 'A Table for the true dividing of the meridians in the sea Chart' (as his Table of Latitudes was called in the 1599 edition of his *Certaine Errors in Navigation*) is indicated by the following, taken from the third page of his tables of 1599 :—

1 Col.		2 Col.	1 Col.		2 Col.	1 Col.		2 Col.
De.	Mi.		De.	Mi.		De.	Mi.	
30	10	18999	35	10	22565	40	10	26358
30	20	19115	35	20	22688	40	20	26489
:	:	:	:	:	:	:	:	:
31	50	20166	36	50	23802	41	50	27683
32	0	20284	37	0	23927	42	0	27818
:	:	:	:	:	:	:	:	:

The base of the system of logarithms which Benjamin Martin attributes to Wright may be obtained thus: take $\theta = 42^\circ$, then $\frac{90^\circ - 42^\circ}{2} = 24^\circ$, $\log_{10}\tan 24^\circ = 9{\cdot}6486 - 10 = -{\cdot}3514$. From the Table of Latitudes we obtain $\log_w \tan 24^\circ = -{\cdot}27818$, where we interpret the tabular numbers as decimals taken negatively. The relation, $\log_{10}\tan 24^\circ = \log_w \tan 24^\circ \,.\, \log_{10} w$, yields $\log_{10} w = 1{\cdot}263$, and $w = 18{\cdot}33\ldots$, which is the base of the system of logarithms alleged to have been invented by Wright.

Benjamin Martin speaks of nautical logarithms in a book which he published thirteen years earlier than his *System of Logarithms*, namely, in his *New and Comprehensive System of Mathematical Institutions*, London, 1759, vol. i, p. 277. But in this earlier publication we failed to find any statement claiming for Wright priority of invention.

[2] Edmund Halley, 'Analogy of Logarithmick Tangents to the Meridian Line,' in the *Philosophical Transactions of London*, No. 219, 1695-6 (published 1698), p. 202.

logarithmic tangents of half the complement of the latitude. He published his discovery in an *Addition* to Norwood's *Epitome of Navigation.* The proof of the truth of this proposition seemed so difficult to Nicolaus Mercator,[1] the author of the *Logarithmotechnia*, that he proposed to 'wager a good sum of Money, against whoso would fairly undertake it, that he should not demonstrate either, that it was true or false.' John Collins, who was in correspondence with the eminent mathematicians of the age, 'did excite them to this enquiry.' On 28th February 1665 John Collins wrote to John Wallis[2]:

'. . . that the Meridian line of Mercator's chart should seem (as it doth) to be the same with the logarithm tangents, (viz. that the adding of natural secants should constitute a logarithm tangent, though to an unwonted ratio,) is Mysterium aliquod grande proposed long since to Mr Briggs and Mr Gunter, but not approved or disproved. See my Plain Sailing, herewith sent, the Navigation part, pp. 117, 44, 45, 60, 61, 62. And admitting that the meridian line were the same with a log. tangent, it would not want a geometrical way to describe it, by such curves as pure geometry will reject.'

It was James Gregory[3] who first demonstrated that Henry Bond's observation was correct, viz. that Wright's Table of Latitudes may be interpreted as being a table of logarithms. Isaac Barrow[3] and John Wallis[3] published discussions of this subject, but the final simplified presentation came from Edmund Halley in the *Philosophical Transactions.* It was about a century after this proof that Benjamin Martin claimed Wright had invented logarithms. It is easy to see that the judicial historian must reject the claim made

[1] Nicolaus Mercator, 'Two Problems in Navigation proposed,' in the *Philosophical Transactions of London,* No. 13, vol. i, 1666, p. 215.

[2] Rigaud, *Correspondence of Scientific Men of the Seventeenth Century,* Oxford, vol. ii, 1841, p. 461. See also pp. 459, 485.

[3] See footnote 2, page 97.

by Martin. Wright did not use his tables as a means of shortening computation by substituting for multiplication, division, involution, and evolution respectively the operations of addition, subtraction, multiplication, and division. The idea of a logarithm was foreign to his mind. He meant simply to construct a table which could be used in correcting certain errors in navigation. As John Collins[1] expressed it, Wright 'happened upon the logarithmes and he did not know it,' 'he made a table of logarithmes . . . before logarithmes were invented and printed, but did not know he had donne it.'

That such a table should turn out to be a table of logarithms is not as strange as it might seem. Any set of numbers in arithmetical progression placed parallel with a set of positive numbers in geometrical progression defines some system of logarithms. The terms in arithmetical progression in Wright's table were the enlarged meridional distances; the terms of the geometrical progression were the tangents of half the complementary latitudes. Wright himself never claimed any part in the invention of logarithms. It is clear that his name must be eliminated from the list of inventors of logarithms. When he saw Napier's book of 1614 he recognised its value, and proceeded at once to translate it from Latin into English. Thereby he helped in spreading the knowledge of the new invention.

If the episode in the history of logarithms introduced by Benjamin Martin discloses a lack of good judgment and of rigour in the treatment of historic data, what shall we say of the episode in which the Danish astronomer, Christian Longomontanus, innocently figures as the inventor of logarithms? The whole story rests upon a gossipy and obviously inaccurate tale of Anthony Wood,[2] to which we shall refer again later, and which runs as follows :—

[1] *Aubrey's Brief Lives* (*Edition* A. Clark), Oxford, vol. ii, pp. 315, 316.

[2] Anthony A. Wood, *Athenae Oxonienses* (*Edition* P. Bliss), London, vol. ii, 1815, pp. 491, 492. The claims of Longomontanus are discussed by Thomas Smith in the article 'Commentariolus de Vita et Studiis H. Briggii' in his *Vitae quorundam*

'It must be now known, that one Dr Craig a Scotch man . . . coming out of Denmark into his own country, called upon Joh. Neper baron of Marcheston near Edinburgh, and told him among other discourses of a new invention in Denmark (by Longomontanus as 'tis said) to save the tedious multiplication and division in astronomical calculations. Neper being solicitous to know farther of him concerning this matter, he could give no other account of it, than that it was by proportionable numbers. Which hint Neper taking, he desired him at his return to call upon him again. Craig, after some weeks had passed, did so, and Neper then shew'd him a rude draught of what he called, *Canon mirabilis Logarithmorum*. Which draught, with some alterations, he printing in 1614, it came forthwith into the hands of our author Briggs, and into those of Will. Oughtred, from whom the relation of this matter came.'

This 'new invention in Denmark (by Longomontanus as 'tis said) to save the tedious multiplication and division,' involving 'proportionable numbers,' might apply to logarithms, or it might apply to trigonometric formulas such as $2 \sin x \,.\, \sin y = \cos(x-y) - \cos(x+y)$, which were being studied at the time. Which of these two can it be? Longomontanus survived Napier twenty-nine years, without ever claiming any right to the invention of logarithms; his published works are said to contain no traces of logarithms. On the other hand, he was interested in simplified trigonometric computation, and he figures in the development of prosthaphæresis.[1] It is perfectly plain that Longomontanus

eruditiss. et ill. Virorum, London, 1707, where he arrives at the conclusion: 'Inventum hoc prorsus mirabile coelesti ingenio Neperi unicè debetur.' Anthony A. Wood and Thomas Smith are both quoted in the article 'Briggs (Henri)' in the *Supplement . . . au Dictionnaire . . . de Mr. Pierre Bayle.* Par Jacques George de Chaufepié, Tome ii, a Amerstam, a La Haye, 1750. Montucla, in his *Histoire des mathématiques*, Tome ii, a Paris, 1758, p. 11, makes reference to this article in this *Dictionnaire*.

[1] David Stewart, Earl of Buchan, and Walter Minto, *An Account of the Life, Writings, and Inventions of John Napier of Merchiston*, Perth, 1778, p. 52.

has no claim whatever as an inventor of logarithms. The historical research which attributed logarithms to him is of the same type as that of the archæologist who argued that the old Babylonians knew telegraphy, since pieces of wire had been found in Babylonian ruins; or of the Egyptian archæologist who replied that in Egyptian excavations wire had *not* been found, indicating that the ancient Egyptians knew *wireless* telegraphy.

It still remains for us to consider the rights of priority of the Swiss Joost Bürgi, who published a table of logarithms [1] in 1620, six years after Napier's publication of his *Descriptio.* The facts of the case as we possess them now have been in print many years, yet most diverse conclusions are drawn by historians of different nationalities. As long as national prejudice seems to figure somewhat in this matter, the work of the impartial historian is unfinished. The known facts are about as follows:—

Napier antedates Bürgi by six years in the date of publication: Napier published in 1614, Bürgi in 1620. Arago [2] stated long ago 'that according to a rule which the principal academies of Europe have solemnly sanctioned, and from which the historian of the sciences dares not deviate without falling into arbitrary assumptions and confusion,' the date of publication is the determining factor in settling priority. Nevertheless, it is true that if an invention is made earlier by one of two men, but published sooner by the other, then the first man attains a certain moral right, in the face of which an arbitrary rule like the one stated by Arago does not always secure undisputed

[1] *Arithmetische und Geometrische Progress-Tabulen sambt gründlichem Unterricht wie solche nützlich in allerley Rechnungen zu gebrauchen, und verstanden werden soll. J.B.* Gedruckt in der alten Stadt Prag bey Paul Sessen der Löblichen Universität Buchdruckern im Jahr 1620.

The promised 'Unterricht' was omitted; it was first published by Gieswald under the title, *Justus Byrg als Mathematiker und deren Einleitung in die Logarithmen*, Danzig, 1856; it is reprinted in *Grunert's Archiv*, 1856, Bd. 26, p. 319.

[2] F. Arago, *op. cit.*, p. 383.

assent. The attempt to ascertain the dates of the inventions of logarithms has not been altogether successful, and has left considerable uncertainty. It has brought a fresh illustration of the wisdom of using Arago's rule in writing the history of science.

In the case of Bürgi the witnesses upon which his claim rests are Benjamin Bramer, who was Bürgi's brother-in-law, and Kepler. Bramer, who lived with Bürgi during the period 1603-1611, to be initiated by Bürgi into the sciences, says [1] that his brother-in-law computed his table in that period. According to this testimony, Bürgi had worked out his logarithms certainly as early as 1611, and possibly as early as 1603. Kepler's statement is indefinite as to dates. He was in friendly touch with Bürgi; he states in the introduction to his Rudolphine Tables [2] that Bürgi had proceeded to the preparation of his logarithmic tables many years before the publication of Napier's *Descriptio.* Kepler blames Bürgi for neglecting to proceed to a prompt publication of his logarithms, and thereby forfeiting priority over Napier. Referring to the apices logistici (the ′ ″ ‴ used in the sexagesimal system) Kepler uses these words: 'The apices logistici led Justus Byrgius on the way to these

[1] Benjamin Bramer, *Beschreibung Eines sehr leichten Perspektiv und grundreissunden Instrumentes auff einem Stande* u.s.w., Cassel, 1630, p. 5. Johannes Tropfke, in his *Geschichte der Elementar-Mathematik,* Band ii, Leipzig, 1903, p. 145, footnote 589, copies the following passage taken from Bramer's work:—'Auff diesem Fundament hat mein lieber Schwager und Präceptor Jobst Burgi vor zwantzig vnd mehr Jahren eine schöne progress-tabul mit ihren Differentzen von 10 zu 10 in 9 ziffern calculiert auch zu Prag ohne bericht in Anno 1620 drucken lassen. Vnd ist also die Invention der Logarithmen nicht dess Neperi, sondern von gedachtem Burgi (wie solches vielen wissend vnd ihm auch Herr Keplerus zeugniss giebt) lange zuvor erfunden.'

[2] Kepler's Rudolphine Tables, published in 1627 [*Tab. Rud.,* cap. iii, p. 11], are found in Kepler's *Gesammelte Werke,* ed. Frisch, Frankfort, 1861-71. See Bd. ii, p. 834; Bd. vii, p. 298. Kepler says: 'Sin optabile tibi est, ex ipso logarithmi characteristico principio arguere speciem logisticam numeri, cui assignatur logarithmus, ecce tibi apices logisticae antiquae, qui praestant hoc longe commodius: qui etiam apices logistici Justo Byrgio multis annis ante editionem Neperianam viam praeiverunt ad hos ipsissimos logarithmos. Etsi homo cunctator et secretorum suorum custos foetum in partu destituit, non ad usus publicos educavit.'

very logarithms many years before Napier's system appeared; but being a hesitating man, and very uncommunicative, instead of rearing up his child for the public benefit, he deserted it in its birth.' What is established with certainty by the testimony of Bramer and Kepler is that Bürgi invented logarithms independently of Napier, and that the *Progress Tabulen* were computed some time between 1603 and 1611, before Napier's publication of logarithms. It is to be noted that the statements of Bramer and Kepler are not sufficient to establish priority of invention for Bürgi.

It is unreasonable to suppose that John Napier's logarithms of 1614 were the fruit of instantaneous development. That Napier had thought out his logarithms many years before their publication follows from his own words and from the statement of his son Robert. In the preface to his *Rabdologia*, Napier refers to his canon of logarithms, 'a me tempore elaboratum,' or in translation, 'for a long period elaborated by me.' This preface was written in 1617, shortly before his death. His son Robert wrote in 1619 a preface to John Napier's *Logarithmorum canonis constructio*, containing a passage which in Macdonald's translation [1] is as follows:—'You have then (kind reader) in this little book most amply unfolded the theory of the construction of logarithms (here called by him artificial numbers, for he had his treatise written out beside him several years before the word Logarithm was invented,) in which their nature, characteristics, and various relations to their natural numbers, are clearly demonstrated.'

There is circumstantial evidence indicating that John Napier worked on logarithms as early as 1594. The evidence consists of a statement in one of Kepler's letters and of the passage from Anthony Wood which we quoted above.

[1] *The Construction of the Wonderful Canon of Logarithms by John Napier, Baron of Merchiston,* translated from Latin into English with Notes, etc., by William Rae Macdonald, F.F.A., Edinburgh and London, 1889.

Kepler's letter,[1] written in 1624 to his friend Cugerus, comments upon the illustrious promoters of trigonometry, and then touches upon logarithms in a sentence which in translation reads as follows:—'But nothing, in my mind, surpasses the method of Napier, although a certain Scotchman, even in the year 1594, held out some promise of the wonderful Canon in a letter to Tycho.' That this correspondent was Dr John Craig, mentioned by Anthony Wood, and that the invention hinted at was the logarithms of Napier himself, is argued with great force by Mark Napier.[2] The assertions in Kepler's letter find some corroboration in the passage from Anthony Wood. As is well known, Kepler worked in Tycho Brahe's observatory at one time, hence was in a position to secure information from Brahe directly.

There is still another and less agreeable chapter in the history of the efforts to settle the rights of priority in the invention of logarithms. There have been those who hinted that Napier borrowed his logarithms from Bürgi, or that Bürgi borrowed his logarithms from Napier. It is a happy circumstance that not a particle of evidence worthy of serious consideration has been advanced by any one in support of these charges.

What, then, are the facts? Our inquiry shows:

(1) That John Napier enjoys the all-important right of priority of publication;

[1] *Kepleri epistolae.*

In the original Latin the passage quoted is as follows:—'Nihil autem supra Neperianam rationem esse puto; etsi quidem, Scotus quidam, literis ad Tychonem, anno 1594, scriptis jam spem fecit Canonis illius Mirifici.'

[2] Mark Napier, *Memoirs of John Napier of Merchiston, his Lineage, Life, and Times, with a History of the Invention of Logarithms,* Edinburgh and London, 1834, p. 364 ff.

De Arte Logistica Joannis Naperi Merchistonii Baronis Libri qui supersunt. Impressum Edinburgi, 1839, p. xxvii ff.

See also J. W. L. Glaisher's article 'Logarithm' in the *Encyclopædia Britannica,* 11th ed.

(2) That Joost Bürgi is entitled to the honour of independent invention;

(3) That Joost Bürgi constructed his table some time between 1603 and 1611, and that John Napier worked on logarithms probably as early as 1594—that, therefore, Napier began working on logarithms probably much earlier than Bürgi.

In view of these facts it follows that the majority of statements found in histories and cyclopædias are incorrect. This condition of things is due mainly to a failure to secure acquaintance with the facts on both sides of the question. Mark Napier's *Memoirs of John Napier* has been little known on the Continent; that Bürgi published tables in 1620 and that Gieswald published Bürgi's explanations of those tables [1] in 1856 have been little known in Great Britain. The fact that the exhibition at this Napier Tercentenary Celebration includes publications not only about Napier, but also about Bürgi, shows that we are beginning to discard provincialism and are in a new era of fairness and justice to all, based upon a broad historical outlook.

Going back to the publications of the past, we find that reasonably correct statements are made by the French historians Bossut [2] and Montucla,[3] the German historians Karl Fink [4] and J. C. Poggendorff,[5] the Danish historian H. G. Zeuthen,[6] the English historian W. W. Rouse Ball,[7]

[1] See footnote 1 on page 101 and 2 on page 104.

[2] John Bossut, *General History of Mathematics*. Translated from the French. London, 1803, pp. 211, 215.

[3] J. F. Montucla, *Histoire des mathématiques*, Tome 2, à Paris, Au vii, pp. 9-12.

[4] Karl Fink, *History of Mathematics*. An authorised translation by W. W. Beman and D. E. Smith, Chicago, 1900, p. 290.

[5] J. C. Poggendorff, *Geschichte der Physik*, Leipzig, 1879, p. 611.

[6] H. G. Zeuthen, *Geschichte der Mathematik im XVI und XVII Jahrhundert*. Deutsche Ausg. v. Raphael Meyer, Leipzig, 1903, pp. 21, 132, 133.

[7] W. W. Rouse Ball, *A Short Account of the History of Mathematics*, 5th ed., London, 1912, p. 196.

the *Edinburgh Encyclopædia*,[1] Klügel's *Mathematisches Wörterbuch*,[2] and Brockhaus' *Conversations-Lexikon*.[3] Montucla remarked long before Arago that priority of publication is recognised by the tribunal of public opinion as fixing priority of invention. Montucla thinks Bramer's claim wide of the mark—the claim that Bürgi deserves priority because he had worked out his logarithms prior to the date of Napier's first publication (1614)—for Napier must have had (says Montucla) the theory of logarithms thought out before he entered upon the long and tedious computation of his tables.

It would seem that Bürgi's name was little known in Great Britain before the latter part of the eighteenth century. John Wallis[4] does not mention Bürgi in his *Treatise of Algebra*. John Keill,[5] professor of astronomy at Oxford, said in an appendix to his edition of Commandine's *Euclid* : 'And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet Neper is universally allowed the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.' Similar remarks occur in George Costard's[6] *History of Astronomy*. Some writers are decidedly unfair to Bürgi. The French historian Marie[7] states that it is difficult to determine whether Bürgi worked independently of Napier; Marie goes apparently on the assumption that Bürgi must be considered guilty unless he can be proved innocent.

[1] *The Edinburgh Encyclopædia*, by David Brewster, Philadelphia, 1832, arts. 'Napier' and 'Logarithm.'

[2] Georg Simon Klügel, *Mathematisches Wörterbuch*, 1. Abtheilung, Die Reine Mathematik, 4. Theil von Q bis S, Leipzig, 1823, art. 'Logarithmus,' p. 531.

[3] Brockhaus' *Conversations-Lexikon*, Leipzig, 1885, art. 'Logarithmus.'

[4] John Wallis, *Treatise of Algebra*, London, 1685, chap. xii, p. 55.

[5] John Keill, *Euclid's Elements of Geometry from the Latin Translation of Commandine*, etc., edited by Samuel Cunn, London, 1759, p. 326.

[6] George Costard, *History of Astronomy*, London, 1767, p. 160.

[7] Maximilian Marie, *Histoire des Sciences mathématiques et physiques*, Tome iii, Paris, 1884, p. 85.

The fourth to the eighth editions (inclusive) of the *Encyclopædia Britannica,*[1] in the article 'Logarithm,' declare Napier 'most probably the only inventor.' Mark Napier[2] declares that 'the value of Byrgius's share of any honour in the matter may be expressed by that ghostly symbol which is the soul of the Arabic notation, 0,' and that 'viewed[3] in every light, the claim for Byrgius is either nonsense or roguery.' This judgment sounds harsh, but there are extenuating circumstances; apparently Mark Napier did not know that Bürgi had ever actually published a table of logarithms. Nor was the publication of such a table known to Biot,[4] who prepared a lengthy review of Mark Napier's biography of his great ancestor.

Other writers are equally unfair to Napier. The Swiss astronomer, Rudolf Wolf,[5] in 1877 gives expression to the possibility that Napier may have received information of Bürgi's research through Longomontanus. This absurd claim is based on the well-known quotation from Anthony Wood, which figures here for the third time in discussions of priority. In 1890 Wolf[6] is more considerate towards Napier, for while he still asserts (without proof) that Bürgi computed logarithms before Napier, he declares as wholly unfounded a claim put forth by Jacomy Régnier[7] in 1855, to the effect that Napier borrowed the logarithms from Bürgi. Delambre's conjecture[8] that Napier may have gotten his logarithms from Bürgi through N. R. Dithmarus, in

[1] *Encyclopædia Britannica*, 4th ed., Edinburgh, 1810; 7th ed., Edinburgh, 1842; 8th ed., Boston, 1857. Article 'Logarithm.'

[2] Mark Napier, *Memoirs*, p. 397.

[3] Mark Napier, *ibid.*, p. 398, footnote.

[4] J. B. Biot, *Journal des Savants*, 1835, pp. 151-162, 257-270.

[5] R. Wolf, *Geschichte der Astronomie*, München, 1877, p. 349.

[6] R. Wolf, *Handbuch der Astronomie*, Erster Halbband, Zürich, 1890, pp. 68-70.

[7] Jacomy Régnier, *Histoire des nombres et de la numération mécanique*, Paris, 1855.

[8] Delambre, *Histoire de l'astronomie moderne*, Tome i, Paris, 1821, pp. 291, 313, 565.

which event Bürgi antedated Napier by twenty-six years, is even less worthy of consideration. Gieswald [1] and the German historians A. G. Kästner,[2] C. I. Gerhardt,[3] Moritz Cantor,[4] J. Tropfke,[5] A. v. Braunmühl,[6] and the French *La Grande Encyclopédie*,[7] set the time of Bürgi's computation of his *Progress Tabulen* prior to Napier's date of publication (1614), but make no effort to ascertain how long before 1614 Napier himself had been working on logarithms; in other words, they compare Bürgi's supposed date of invention with Napier's date of publication, and therefrom do not conclude, as they legitimately could, that Bürgi was an independent inventor, but they conclude, as they cannot legitimately do, that Bürgi's invention was prior to Napier's, or that Bürgi very probably lost priority simply because of failure to publish his logarithms as soon as invented by him.

Three German writers [8] go further yet; without reserve and without proof they attribute the earliest invention to Bürgi. One of those writers declared within the past

[1] Gieswald, *op. cit.* [footnote 1 on page 101], Danzig, 1856. Gieswald says on pages 14 and 15: '. . . es ist wahrscheinlich nicht nur die *Rhabdologia*, sondern auch die erste Aufstellung von Logarithmentafeln das Werk eines Deutschen.—Allerdings müssen wir auch die Verdienste *Nepers*, der selbstständig die erfasste Idee weiter verfolgte und zum Ziele führte, rühmlich anerkennen. . . .'

[2] Abraham Gotthelf Kästner, *Geschichte der Mathematik*, Göttingen, vol. ii, 1797, p. 375; vol. iii, 1799, p. 14.

[3] C. I. Gerhardt, *Geschichte der Mathematik in Deutschland*, München, 1877, pp. 114, 118.

[4] Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, Bd. ii, 1892, p. 662. 'Bevor wir indessen von dieser handeln [Napier's *Descriptio*], ist es wohl richtiger, von einer später veröffentlichten, doch mit grosser Wahrscheinlichkeit früher entstandenen verwandten Leistung zu berichten, von den Progress-Tabulen des Jobst Bürgi.'

[5] J. Tropfke, *op. cit.*, Bd. 2, p. 145.

[6] A. v. Braunmühl, *Geschichte der Trigonometrie*, Leipzig, 1903, Teil 2, p. 2.

[7] *La Grande Encyclopédie* (Berthelot), Paris, art. 'Logarithmes.'

[8] Ambros Sturm, *Geschichte der Mathematik*, Leipzig, 1904, p. 102. E. Gerland, *Geschichte der Physik*, Leipzig, 1892, p. 73. Max Simon, 'Die Dreihundertjahrfeier der Logarithmentafel,' in *Oesterreichische polytechnische Zeitschrift*, August 15, 1913, p. 159.

twelve months that Bürgi invented logarithms 'at least five years before Napier.' Nothing is gained by such distortions; they have no place in impartial history. The facts as they are known to-day assign to Napier the glory of a star of the first magnitude as the inventor of logarithms who first gave them to the world, and to Bürgi the glory of a star of lesser magnitude, but shining with an independent light.

In conclusion, it may not be out of place to note that Scotland and Switzerland, the countries which produced Napier and Bürgi, have much in common. Both are small countries, yet they contributed much to the progress of the exact sciences. Scotland produced Napier and the authors of Thomson and Tait's *Natural Philosophy*; Switzerland produced the Bernoullis and Euler. In past centuries both countries made heroic struggles for civil liberty. Scotland had her William Wallace and Robert Bruce; Switzerland had her Arnold von Winkelried and Wilhelm Tell. May the future of both countries be as brilliant as has been their past.

NAPIER'S LOGARITHMS AND THE CHANGE TO BRIGGS'S LOGARITHMS

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Of Napier's two works on logarithms, the *Descriptio* of 1614 and the *Constructio* of 1619, the latter has been excellently translated and edited by Mr W. Rae Macdonald, and is readily accessible. But the *Descriptio*, in which the logarithms are explained and applied, is now a somewhat rare book, and as the peculiarities of Napier's logarithms can be most fully understood from the rules and examples to be found in his first published treatise, it seems to be not inappropriate that an account of his system should be given in connection with the tercentenary celebrations. In the following sketch I will, as far as possible, use the words of Wright's translation of the *Descriptio*,[1] but the logarithms and numbers will be taken from the Latin edition, as Wright contracted the logarithms by one figure. I hope the account will be sufficiently full to enable the reader to understand what Napier's logarithms were and what were their rules of operation. In the course of the work the defects naturally attaching to a new instrument will become plain, and the reasons for the change to another system will be more obvious. I give a discussion of the change to Briggs's logarithms as I believe that the nature of that change is not generally understood. I have no wish to revive ancient controversies, but I think that no harm can be done by stating as fairly as I can what seem to me to be the facts

[1] The first and second chapters of Wright's translation are reproduced in facsimile in Plates I to VI in this volume.

of the case. My investigations lead me to believe that Napier has not quite received full credit for his contribution to the change.

The text of the *Descriptio* consists of two books. The first book discusses in five chapters the Definitions and working Rules of Logarithms; the second book, containing six chapters, applies the logarithms to various problems of trigonometry, plane and spherical. The first edition is a small-sized quarto, containing fifty-seven pages of explanatory matter and ninety pages of tables.

Napier bases his system on the correlation of two types of motion, and begins the *Descriptio* with two definitions (see Plates I and II).

Definition 1.—A line is said to increase equally when the point describing the same goeth forward equal spaces in equal times or moments.

Corollary.—By this increasing, quantities equally differing must needs be produced in times equally differing.

Definition 2.—A line is said to decrease proportionally into a shorter when the point describing the same in equal times cutteth off parts continually of the same proportion to the lines from which they are cut off.

Corollary.—By this decrease in equal moments or times there must also be left proportional lines of the same proportion.

In the *Constructio* these lines are said to increase arithmetically and to decrease geometrically respectively.

The definitions are illustrated by Fig. 1.

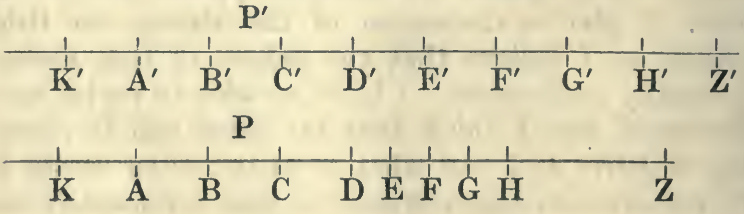

Fig. 1

AZ is a given line which is to be decreased proportionally by the motion of the point P; A′Z′ is a line of unlimited length which is to be increased equally by the motion of the point P′. Let B, C, D . . . be taken on AZ such that

$$AB : AZ = BC : BZ = CD : CZ = \dots$$

and let B′, C′, D′ . . . be taken on A′Z′ so that

$$A'B' = B'C' = C'D' = \dots$$

If the points P and P′ describe the lines AZ and A′Z′ in such a way that they pass over the segments AB, BC, CD, . . . and A′B′, B′C′, C′D′, . . . in equal times, then the line PZ decreases proportionally or geometrically, and the line A′P′ increases equally or arithmetically.

Obviously

$$AZ : BZ = BZ : CZ = CZ : DZ = \dots$$

so that AZ, BZ, CZ, DZ . . . are in continued proportion.

After stating (see Plate III) that surd quantities may be expressed *quam proxime* by rational approximations, Napier defines 'equal-timed motions' as 'those which are made together and in the same time,' and states the following postulate and definition:—'Seeing that there may be a slower and a swifter motion given than any motion it shall necessarily follow that there may be a motion given of equal swiftness to any motion (which we define to be neither swifter nor slower).' Napier required for the full development of his system a precise conception of a variable velocity, and he bases his treatment on this postulate; but it is in the *Constructio* (arts. 25, 28, etc.) rather than in the *Descriptio* that he applies it, and even then in a somewhat sketchy way, though he thoroughly grasped the special theorem in variable velocity needed for his work.

Before giving Napier's definition of a logarithm I may note that in his day $\sin\theta$, $\tan\theta$, $\sec\theta$ were lines, not ratios; if θ is the angle at the centre of a circle of radius r the lines $r\sin\theta$, $r\tan\theta$, $r\sec\theta$ are Napier's $\sin\theta$, $\tan\theta$, $\sec\theta$. The radius r is called 'the whole sine'; both in the *Descriptio* and in the *Constructio* $r=10^7$.

Definition of a Logarithm

'The logarithm of any sine is a number very nearly expressing the line which increased equally in the meantime while the line of the whole sine decreased proportionally into that sine, both motions being equal-timed and the beginning equally swift' (see Plate III).

In Fig. 1 let $AZ=r$, the whole sine, and let P, P′ be corresponding positions of the moving points; these are supposed to start at the same instant with the same velocity from A and A′ and to move for the same time, P geometrically and P′ arithmetically. The number that measures A′P′ is the logarithm of the number that measures PZ; or, more briefly,

$$A'P' = \log PZ.$$

Thus the logarithms of BZ, CZ, DZ . . . are A′B′, A′C′, A′D′ . . . respectively.

It follows from the definition that the logarithm of r, the whole sine, is zero, and this is a peculiarity of Napier's system. He states explicitly, however, that this choice of the number whose logarithm is zero is quite arbitrary; he chose r with a view to simplicity in trigonometric applications. At a later date, as will be pointed out below, he saw that it would be better to choose zero as the logarithm of unity, and he was, I believe, the first who is known to have suggested that choice.

Again, if ZA is produced backwards and if P moves backwards from A so that PZ increases proportionally, then the logarithm of a number, such as KZ, greater than AZ will be negative, or, in Napier's language, 'defective' or 'wanting.' Thus $\log x$ is positive or negative according as x is less or greater than the whole sine.

In the *Constructio*, which was written some years before the publication of the *Descriptio*, the phrase 'artificial number' is used instead of 'logarithm,' that word being of later invention (see Robert Napier's preface to the *Constructio*). The word 'logarithm' is usually explained as 'measure of ratio' or as 'number of ratios'; a logarithm in the latter mode of explanation expresses 'the number of *ratiunculae* contained in any ratio, or into which it may be divided, the number of the like equal *ratiunculae* contained in some one ratio, as of 10 to 1, being supposed given' (Hutton's *Tracts*, vol. i, p. 406, or *Scriptores Logarithmici*, vol. vi, p. 635). I think there are more grounds than one for rejecting this derivation. The more likely derivation seems to me to be simply this, that logarithm means 'ratio-number' or 'number associated with ratio,' as suggested by Briggs, *Arithmetica Logarithmica*, chap. i, where he states: 'They seem to have been called logarithms by their illustrious inventor because they exhibit to us numbers which always preserve the same ratio to one another.' If any one knew Napier's work thoroughly it was Briggs, and any suggestion from him deserves the greatest weight. Napier alone knew the derivation of the word and dogmatism in the matter is now out of place, but I have myself little or no doubt that logarithm means merely 'ratio-number' or 'number associated with ratio.' The term is more descriptive than the vague phrase 'artificial number,' and fits in very well with the opening sentences of the *Constructio*.

The second chapter of the *Descriptio* (see Plates IV and V) deals with the Propositions of Logarithms. The fundamental theorem is

Proposition 1.—The logarithms of proportional numbers and quantities are equally differing.

Napier illustrates rather than proves this proposition, but he thoroughly understood it. He points out (Fig. 1) that the four lines BZ, DZ, FZ, HZ form a proportion

$$BZ : DZ = FZ : HZ;$$

the logarithms of these lines are represented by A′B′, A′D′, A′F′, A′H′, and

$$A'D' - A'B' = B'D' = F'H' = A'H' - A'F';$$

that is, $\log DZ - \log BZ = \log HZ - \log FZ.$

The best discussion of logarithms on Napier's lines is that of Maclaurin (*Fluxions*, Book I, chap. vi), and is well worth careful study even now. A fuller statement of Napier's reasoning is suggested by Maclaurin's treatment.

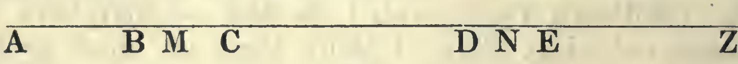

Fig. 2

Let BC, DE (Fig. 2) be described by P in equal times, and let BM, DN be described by P in equal parts of those times; then by the definition of the motion of P

$$BC : BZ = DE : DZ, \quad BM : BZ = DN : DZ,$$

and therefore $BM : DN = BZ : DZ,$

so that $BM : DN = BZ : DZ = CZ : EZ = MZ : NZ,$

the three last ratios being equal to one another.

Hence the distances described by P in equal times are in the same ratio as the distances of P from Z at the beginning, at the end, or at any corresponding intervals of those times. If BZ : CZ and DZ : EZ are any two equal ratios, the distances BC and DE will be described in equal times since these distances are proportional to BZ and DZ and also to CZ and EZ; therefore also the corresponding distances B′C′ and D′E′ will be described by P′ in equal times and will thus be equal. That is,

if $BZ : CZ = DZ : EZ$

then $\log CZ - \log BZ = \log EZ - \log DZ.$

From Proposition 1 the others readily follow.

Proposition 2.—If $a : b = b : c$, $\log c = 2 \log b - \log a$.

For, by Prop. 1, $\log b - \log a = \log c - \log b$.

Proposition 3.—If $a : b = b : c$, $2 \log b = \log a + \log c$.
Proposition 4.—If $a : b = c : d$, $\log d = \log b + \log c - \log a$.
Proposition 5.—If $a : b = c : d$, $\log b + \log c = \log a + \log d$.
Proposition 6.—If $a : b = b : c = c : d$,

$$3 \log b = 2 \log a + \log d,$$
$$3 \log c = \log a + 2 \log d.$$

These are all the rules that are explicitly stated, and it should be noticed that they are associated with proportions; in other words, before the logarithms are applied a proportion has to be formed, one of its terms being the number whose value is required.

The third chapter contains a description of Napier's Table of Logarithms. The Table gives the sines, and the logarithms of the sines and of the tangents of all angles from 0° to 90° at intervals of one minute; it is arranged semi-quadrantally, so that the logarithms of the sine and the cosine of an angle appear on the same line, their difference being given in the column of *Differentiae*, which thus forms a table of logarithms of tangents. The numbers therefore do not proceed at equal intervals; the sines are tabulated for angles at intervals of one minute, so that the table is inconvenient as a table of numbers.

Again, to find the logarithm of a number greater than the whole sine, or 10^7, it is first necessary to consult a table of natural tangents or natural secants and ascertain the angle which has the given number as its tangent or secant. For example, 30776835 is tan 72°; from Napier's Table the logarithm of this tangent is −11241768, which is therefore log 30776835.

The fourth chapter, 'Of the Use of the Table and of its Numbers,' and the fifth, 'Of the most ample Use of the Logarithms and ready Practice by them,' bring out the peculiarities of Napier's system.

The first point that may be noticed is that the familiar theorems 'the logarithm of a product is the sum of the logarithms of its factors,' 'the logarithm of a fraction is the logarithm of the numerator diminished by the logarithm of the denominator,' and the like, are not generally true. No doubt, by Prop. 4,

$$\log \frac{bc}{a} = \log b + \log c - \log a,$$

but $\log bc$ is not $\log b + \log c$ and $\log \frac{b}{a}$ is not $\log b - \log a$. If we write $x = bc$ and form the proportion

$$x : b = c : 1,$$

then

$$\log x = \log b + \log c - \log 1,$$

and log 1 is not zero. The awkwardness of the choice of r, or the whole sine, as the number whose logarithm is zero makes itself felt whenever the proportion contains unity as one of its terms. We might, of course, put $x = bc/r$ and then

$$\log x = \log b + \log c - \log r = \log b + \log c;$$

we should thus find x and then rx or bc. Similar devices could be applied for obtaining $\sqrt{(abc)}$ and the like, but it is obvious that for such calculations the fact that log 1 is not zero ($\log 1 = 7 \log 10^6$) is a serious drawback to the usefulness of logarithms.

In working with this system one must constantly keep the fundamental proposition in mind, that if two ratios are equal the differences of the logarithms of their terms are equal; or, in algebraic notation,

$$\log ak - \log bk = \log a - \log b.$$

Thus such relations as the following are obtained :—

$$\log ac - \log bd = (\log ac - \log bc) + (\log bc - \log bd),$$
$$= (\log a - \log b) + (\log c - \log d);$$

$$\log (a^2) - \log (b^2) = 2(\log a - \log b),$$

and these are easily extended. The equation

$$\log (a^n) - \log (b^n) = n(\log a - \log b)$$

holds whether n is integral or fractional, positive or negative. The connection between the logarithms of a and $10^n a$ is important.

$$\log a - \log 10a = \log 1 - \log 10 = \log 10^6 = t, \text{ say,}$$

because $1 : 10 = 10^6 : 10^7$ and $\log 10^7 = 0$.

$$\log a - \log 100a = 2(\log 1 - \log 10) = 2t$$

$$\log a - \log (10^n a) = n(\log 1 - \log 10) = nt.$$

Also, $\log a - \log 20a = (\log 1 - \log 2) + t$

$$= \log 5000000 + t.$$

$$\log a - \log 200a = \log 5000000 + 2t,$$

and so on. The value of t as given by Napier is 23025842·34, but is more nearly 23025850·93. (See note, pp. 90-96, of Mr Macdonald's translation of the *Constructio*.) Of course, Napier does not state the relations in the above form, though he uses the specific cases.

If $x = \dfrac{A^a B^b C^c \ldots}{D^d E^e F^f \ldots}$ it is easily seen that

$$\log x = \Sigma a(\log A - \log 1) - \Sigma d(\log D - \log 1) + \log 1$$

$$= \Sigma a \log A - \Sigma d \log D - (\Sigma a - \Sigma d - 1) \log 1$$

and $\log 1 = 7 \log 10^6 = 7t$.

The method of finding logarithms of numbers not given exactly in the Table has next to be considered, and it must be admitted that there is some truth in the remark that occurs in the appendix to the 1618 edition (or rather, reprint) of Wright's translation of the *Descriptio*, to the effect that Napier's treatment of this matter is 'wondrously perplexed.'

Napier's first example is to find log 137, and to this end he seeks in the tables of natural sines, tangents, and secants 'the number that is most like 137, whether it be tenfold, a hundredfold, a thousandfold, etc.' Among the sines he finds

14544, 136714, 1371564 ;

among the tangents, 13705046, and among the secants, 13703048. The last number is the 'most like' to 137 'provided it be understood that the last five figures are to be deleted.' The logarithm of 13703048 is −3150332, 'which is also taken for the logarithm of 137, it being remembered however that the last five figures are to be deleted, or, *memoriae gratia*, to be noted expressly in this way

$$-3150332-00000.\text{'}$$

Similarly log 232702 is found from the most like number, 23271, which has one digit less; log 23271 is 60631284, and log 232702 is denoted by

$$60631284+0.$$

Napier points out that for many purposes the errors involved in this method are of little moment; for a more perfect way of estimating such logarithms he says that recourse should be had to the method by which the logarithms were constructed, but he gives no account of the method in the *Descriptio*.

A logarithm with cyphers annexed to it is called 'impure'; the rules for addition and subtraction of logarithms, whether pure or impure, are those of algebra. For example, to subtract $-4216+00$ from $5392+0$ is the same as to add $4216-00$ to $5392+0$; the result is $9608-0$.

The determination of the number from its logarithm is sometimes simplified by the following rule, where $t=\log 10^6$ $=23025842$:—'To increase or diminish a logarithm in number, its former value remaining, is to add to it or subtract from it any of the logarithms following

$$t+0,\ 2t+00,\ 3t+000,\ 4t+0000,\ 5t+00000,$$

which signify nothing at all.'

As an example, to $39156-0$ add $t+0$ and there results 23064998, 'which is greater in number but altogether the same in value as $39156-0$.' From the Table 996092 is the number which has 23064998, and therefore also $39156-0$, as its logarithm.

The relation $\log a = \log 10a + t$ shows the foundation for the rule; the addition of t to the logarithm of any number corresponds to the division of that number by 10. Again, the symbol $+0$ annexed to a logarithm is an instruction to multiply the number corresponding to the pure logarithm by 10. Thus the number whose logarithm is $\log b+t+0$

is $(b \div 10) \times 10$, that is, b. Similarly the number whose logarithm is $\log b - t - 0$ is $(b \times 10) \div 10$, that is, b; and so on.

The accurate determination of the logarithm of a number not found exactly in the Table is often a matter of some difficulty; the application of the method of proportional parts or of differences is troublesome because the numbers do not proceed at equal intervals. In the appendix to the 1618 edition of Wright's translation an auxiliary table is given, which simplifies the work considerably; the author of this table is not stated. (Dr Hutton suggests Briggs.) In the translation the whole sine is taken as 10^6, and both numbers and logarithms are given to one figure less than in Napier's table. It is noted that the differences of the logarithms are equal to the differences of the sines until the sine has decreased from 10^6 to about 980000 and the logarithm has increased to about 202000. Rules are given for bringing the sine or the logarithm within the above range, where the use of proportional parts is easy.

To find the logarithm of a given number, the number is first multiplied by an integer a; then, if the product is not within the range, the product is multiplied by $1+\frac{b}{10}$, where b is an integer less than 10; if this second product is not within the range it is multiplied by $1+\frac{c}{100}$, where c is an integer less than 10. The process is simple and effective in many cases. The different stages may be represented thus, x being the given number:

(i) $\log x - \log ax = \log 1 - \log a = d_1$

(ii) $\log ax - \log\left\{ax\left(1+\frac{b}{10}\right)\right\} = \log 10 - \log(10+b) = d_2$

(iii) $\log\left\{ax\left(1+\frac{b}{10}\right)\right\} - \log\left\{ax\left(1+\frac{b}{10}\right)\left(1+\frac{c}{100}\right)\right\}$
$= \log 100 - \log(100+c) = d_3$

so that $\log x = \log\left\{ax\left(1+\frac{b}{10}\right)\left(1+\frac{c}{100}\right)\right\} + d_1 + d_2 + d_3.$

A table is given for d_1 corresponding to the following values of a :—1, 2, . . . 9 ; 10, 20, . . . 90 ; 100, 200, . . . 900 ; and so on up to 100,000, 200,000, . . . 900,000. Tables are also given for d_2 and d_3 corresponding to the values 1, 2, . . . 9 of b and c. The tables serve also, of course, for the converse process of finding the number from the logarithm.

A few examples from Book II of the *Descriptio* may be given. The triangles chosen by Napier are not always of the most suitable kind, and it is rather curious that one of his examples is 'the ambiguous case,' so long a favourite with examiners.

Example 1.—In the triangle ABC, $a=9385$, $c=9384$, $A=90°$; to find C and B.

The proportion required is

$$\sin C : r = c : a$$

where r is the whole sine.

$$\begin{aligned}\log \sin C &= \log c - \log a\\ &= (635870-000)-(634799-000)\\ &= 1071.\end{aligned}$$

Therefore $C=89° \; 9'\tfrac{3}{4}$, $B=50'\tfrac{1}{4}$.

All the logarithms are obtained from the sine column, the logarithms of a and c being given by the most like numbers 9384930 and 9383925 and the angle C (apparently) by proportional parts.

Example 2.—In the triangle ABC, $b=137$, $c=9384$, $A=90°$; to find B.

Here $\tan B : r = b : c$

$$\begin{aligned}\log \tan B &= \log b - \log c\\ &= (42924534-000)-(635870-000)\\ &= 42288664,\end{aligned}$$

from which is obtained $B=0° \; 50' \; 11''$.

It may be noted that in chapter iv Napier found log 137 by the method of the most like number, but he does not use here the value he chose previously as the best ; here he obtains log 137 from the sine 136714. From the table the angle B is more nearly 50′ 5″, though 50′ 11″ is the more correct value; an angle a little greater than 50′ 11″ would have been obtained had the value found in chapter iv for log 137 been used.

Example 3.—a=57955, c=26302, C=26°; to find A.

This is the ambiguous case of our textbooks. Here

$$\sin A : \sin C = a : c$$

$$\begin{aligned} \log a &= 5454707-00 \\ \log \sin C &= 8246889 \\ \hline \text{sum} &= 13701596-00 \\ \log c &= 13354921-00 \\ \hline \text{difference} &= 346675 \\ &= \log \sin A. \end{aligned}$$

'A is 75° (and a little more) if A appear to be an acute angle; otherwise 105° if it appear to be an obtuse angle.'

Example 4.—c=26302, a=57955, b=58892; to find the two 'cases.'

The two 'cases' are the projections of the two sides of a triangle on the base; when the two cases have been found the base-angles are determined from two right-angled triangles. Napier takes b as the base; if D is the projection of B on AC then CD is the greater case, AD is the less, and CE, the difference between CD and AD, is the 'altern base.'

The proportion required is

$$CE : a+c = a-c : b.$$

$$a+c=84257, \quad a-c=31653$$

$$\begin{aligned} \log (a+c) &= 24738819-0 \\ \log (a-c) &= 34529210-0 \\ \text{sum} &= 59268029-00 \\ \log b &= 5293461-00 \\ \text{difference} &= 53974568 \\ &= \log CE. \end{aligned}$$

Hence $\quad CE=45286, \; CD=\frac{1}{2}(CA+CE)=52089$

$\qquad AD=\frac{1}{2}(CA-CE)=6803.$

In these examples the chief difficulty lies in the estimation of the logarithm and of the number when they are not given exactly in the table. A number may have its logarithm determined from a sine, a tangent, or a secant, and the search for the most suitable of these three functions adds considerably to the labour. Different choices do not always lead to concordant results, as pointed out in the case of example 2, though the differences should not always be attributed to the defects of the instrument; in the above case they are due to an unwarranted approximation such as nearly always attaches to the method of 'the most like number.' Some discrepancies are also, I think, due to the imperfections of the table, now known to be caused by some error in the calculations (see the note in Mr Macdonald's translation of the *Constructio*, already referred to). Napier's logarithms as they stand in the *Descriptio* were undoubtedly an enormous advance on the methods of calculation previously in use; yet in their first form their practical application suffered from imperfections that lend some colour of justification to the remarks in the appendix to the 1618 edition of Wright's translation. It is there stated, in reference to the estimation of the logarithms and numbers not found exactly in the Table, that even 'in such places where proportion will perform it, the work for the most part is so manifold (as taking of three differences, then multiplying, and lastly dividing) that the ease which the Book promiseth is oftentimes dearly bought before we can find out the just terms, and the use of the translator's instrument is too much upon conjecture and mechanical.' Of course, interpolations are always troublesome, but we may reasonably apply Napier's own remark in the note at the end of his Table (Mr Macdonald's translation of the *Constructio*, p. 87): 'Nothing is perfect at birth.' The defects, when all is said, do not touch the fundamental conception so luminously expounded and, when all the circumstances are considered, so effectively translated into practice in the first treatise on logarithms. A very short use of the new instrument was certain to bring out its weak points, and probably nobody was more fully aware of the defects and the best method of removing them than Napier himself.

The defects of Napier's logarithms are twofold. In the first place, they are not adapted to the denary scale of numbers since the multiplication or division of a number by 10 corresponds to the subtraction or addition of a very cumbrous number. In the second place, the working rules of logarithms as applied to the calculation of products, quotients, powers, and roots are too complicated; this defect is due to the fact that the logarithm of unity is not zero. The substitution of a power of 10 for the number 23025842, with the consequent change in the logarithms, would partially remove the first defect; but the complete adaptation to the denary scale requires also the choice of the logarithm of unity as zero, which at the same time removes the second defect.

Most of the modern accounts of the change from Napier's logarithms to those of Briggs seem to me to be derived chiefly from Hutton's *History* (Hutton's *Tracts on Mathematical and Philosophical Subjects*, vol. i, tract 20), rather than from independent investigation of the original documents. There is no doubt, I think, that Hutton was not fair to Napier, and a careful examination of the known facts seems therefore to be not unnecessary. I believe that such examination will show that Napier has not received the credit he deserves for the ultimate form that the logarithms assumed, and that Briggs's first proposals have been strangely misinterpreted. Briggs was a mathematician whose services in the promulgation of logarithms can hardly be overestimated, but he would have been the last man to claim any credit that was not strictly due to him, and least of all at Napier's expense. His own account of the matter appears to me to be absolutely clear, and the only difficulty to be explained is Hutton's representation of the case.

The most important document is Briggs's statement in the preface to the *Arithmetica Logarithmica* of 1624, of which the following is a translation :—

'That these logarithms differ from those which that illustrious man, the Baron of Merchiston, published in his *Canon Mirificus* must not surprise you. For I myself, when expounding their doctrine publicly in London to my auditors in Gresham College, remarked that it would be much more convenient that 0 should be kept for the logarithm of the whole sine (as in the *Canon Mirificus*) but that the logarithm of the tenth part of the same whole sine, that is to say, 5 degrees 44 minutes and 21 seconds, should be 10,000,000,000. And concerning that matter I wrote immediately to the author himself; and as soon as the season of the year and the vacation of my public duties of instruction permitted I journeyed to Edinburgh, where, being most hospitably received by him, I lingered for a whole month. But as we talked over the change in the logarithms he said that he had for some time been of the same opinion and had wished to accomplish it; he had however published those he had already prepared until he could construct more convenient ones if his affairs and his health would admit of it. But he was of opinion that the change should be effected in this manner, that 0 should be the logarithm of unity and 10,000,000,000 that of the whole sine; which I could not but admit was by far the most convenient (*longe commodissimum*). So, rejecting those which I had previously prepared, I began at his exhortation to meditate seriously about the calculation of these logarithms; and in the following summer I again journeyed to Edinburgh and showed him the principal part of the logarithms I here submit. I was about to do the same in the third summer also, had it pleased God to spare him to us so long.'

The first thing that strikes one on reading this passage is that Briggs himself believed that Napier had been convinced of the desirability of a change not merely before the receipt of the letter but before the publication of the *Canon Mirificus*; Briggs evidently had no doubt of Napier's good faith in the matter. The second point is that Napier was so fully alive to the nature of the change that he put forward a proposal which Briggs not only accepted but stated to be 'by far the most convenient' (*longe commodissimum*). It is rather strange that the suggestion which Briggs commends so emphatically should be described by Hutton, in a passage which will be quoted presently, as being merely equivalent to a change in the sign but not in the figures of the logarithms. Even such a distinguished and fair-minded critic as Dr Glaisher calls Napier's proposal 'a slight improvement, viz. that the characteristics of numbers greater than unity should be positive and not negative, as Briggs suggested' (*B. A. Report*, 1873, p. 49). Hutton, it may be noted, renders *longe commodissimum* by 'much better.'

Now what exactly was the difference between Napier's logarithms and those suggested in Briggs's letter? The difference was simply that the logarithm of $r/10$, where r denotes the radius or whole sine, should be 10000000000 or 10^{10} instead of, as in Napier's system, the cumbrous number 23025842. The proposal was thus in all essentials the same as that stated by Napier in the 'Admonition' inserted in Book I, chapter iv, of Wright's translation of the *Descriptio*. It has to be particularly observed that it is *the whole sine*, not unity, which Briggs proposed as the number whose logarithm was to be zero; unless this element of the proposal be noted—and Briggs himself calls specific attention to it—it is easy to misinterpret the whole passage. It is probable that Briggs intended to make the whole sine 10^{10}, instead of 10^7 as in the *Descriptio*, and I will assume that r has this value; the particular power of 10 chosen for r does not matter much, but 10^{10} is the best in some respects. Briggs's proposed logarithms were therefore determined by the conditions:

$$\log r = \log 10^{10} = 0 \ldots (1), \log \frac{r}{10} = \log 10^9 = 10^{10} \ldots (2).$$

In modern notation (which lends a deceptive simplicity to the problem), if x is Napier's original and z Briggs's proposed logarithm of the number y, the relations between x, y and z, y are given by the equations

$$y = 10^7 \cdot e^{-\frac{x}{10^7}} \ldots\ldots\ldots\ldots (3),\quad y = 10^{10} \cdot 10^{-\frac{z}{10^{10}}} \ldots\ldots (4);$$

$$\text{or } x = 10^7(7 \log_e 10 - \log_e y) \ldots (3a), z = 10^{10}(10 - \log_{10} y) \ldots (4a).$$

The practical advantage of Briggs's proposed system over that of Napier lies in the simplicity of the relation between the logarithm of a and the logarithms of $10a$, $100a$, . . . $a/10$, $a/100$ . . . ; the addition or subtraction of 23025842 and its multiples is replaced by the addition or subtraction of 10^{10} and its multiples. But the system is quite different from that of the *Arithmetica Logarithmica*, and no mere change of sign would convert one system into the other; only in a restricted meaning of the phrase 'decimal logarithms' could the proposed system be called one of decimal logarithms, and the term 'characteristic' would have quite a different definition from that given to it in the *Arithmetica Logarithmica*. The relation between the logarithms of a and $10^n a$ would be given by the equations

$$\log a - \log (10^n a) = n(\log 1 - \log 10) = n \cdot 10^{10} \ldots\ldots (5),$$

since $\quad \log 1 - \log 10 = \log 10^9 - \log 10^{10} = 10^{10}$,

by equations (1) and (2) above.

The only number to which the term 'characteristic' would seem to be applicable is n, but the value of n gives no information as to the powers of 10 between which either a or $10^n a$ lies; it gives no indication of the size of either number, but merely shows that one is 10^n times the other. It may, however, be noted that the logarithms of unity and the powers of 10 have very simple values; thus if $\tau=10^{10}$

$$\log 1=10\tau,\ \log 10=9\tau,\ \log 100=8\tau,\ \ldots$$
$$\log 0{\cdot}1=11\tau,\ \log 0{\cdot}01=12\tau,\ \log 0{\cdot}001=13\tau,\ \ldots$$

and, in general, $\log(10^n)=(10-n)\tau.$

But the definition of 'characteristic,' if that term were to serve the same purpose as in the *Arithmetica Logarithmica*, would obviously be a little complicated.

(It may be pointed out that if $r=10^p$ and $\tau=10^{10}$ we should have $\log 1=p\tau$, $\log(10^n)=(p-n)\tau$, so that 10 is the simplest value of p.)

The change of sign of n in the last member of (5) is equivalent to the choice of $\log 10r$ instead of $\log(r/10)$ to be 10^{10}, and the effect would simply be to change the sign of all the logarithms; thus we should have

$$\log 1=-10\tau,\ \log 10=-9\tau,\ \log 0{\cdot}1=-11\tau\ \ldots$$

If a number such as 2 be taken, its logarithm on Briggs's proposed system would be 96989700043, while the logarithm of 2×10^{10} would be -3010299957. Obviously if each logarithm of this system were subtracted from 10τ (that is, 10 times Briggs's proposed logarithm of the tenth part of the whole sine), the resulting logarithms would be the ten-figure logarithms of the system developed in the *Arithmetica Logarithmica*. This simplification is no doubt very obvious to those who are familiar with the later system; but it is evident from the fact that Briggs had begun the calculation of a new table before his visit to Napier, that he saw no reason for such a transformation. There is no question of a change of sign; it is simply a case of subtraction from the number 10τ.

Finally, it must be emphasised that in Briggs's suggested system such theorems as

$$\log ab=\log a+\log b,\ \log\frac{a}{b}=\log a-\log b$$

do not hold, any more than in Napier's. The suggested system would unquestionably have been much simpler in various applications than Napier's, but it would have been far from possessing the simplicity and flexibility that are found in the logarithms of the *Arithmetica Logarithmica.*

The vital element in Napier's suggestion was that the logarithm of unity should be zero; the choice of 10^{10} as the logarithm of $\frac{r}{10}$ made the new system 'much more convenient,' but the choice of 0 as the logarithm of unity made the system 'by far the most convenient,' and Briggs states explicitly that this essential improvement came from Napier. Hutton's account is vitiated throughout by the assumption that Briggs's original proposal included the suggestion that the logarithm of unity should be zero, and Dr Glaisher seems to have been misled, in the passage quoted above, by what Hutton says of the change. Hutton quotes the passage from the preface to the *Arithmetica Logarithmica*, and then proceeds (*Tracts*, vol. i, p. 328):

'So that it is plain that Briggs was the inventor of the present scale of logarithms in which 1 is the logarithm of the ratio of 10 to 1, and 2 that of 100 to 1, etc.; and that the share which Napier had in them was only advising Briggs to begin at the lowest number 1, and make the logarithms, or artificial numbers as Napier had also called them, to *increase* with the natural numbers instead of *decreasing*; which made no alteration in the figures that expressed Briggs's logarithms, but only in their affections or signs, changing them from negative to positive; so that Briggs's first logarithms to the numbers in the second column of the annexed tablet would have been as in the first column; but after they were changed, as they are here in the third column; which is a change of no essential difference, as the logarithm of the ratio of 10 to 1, the radix of the natural system of numbers, continues the same; and a change in the logarithm of that ratio being the only circumstance that can essentially alter the system of logarithms, the logarithm of 1 being 0. And the reason why Briggs, after that interview, rejected what he had before done and began anew was probably because he had adapted his new logarithms to the approximate sines of arcs, instead of to the round or integer numbers; and not from their being logarithms of another system, as were those of Napier.'

B	Num.	N
n	$\cdot 01^n$	$-n$
3	·001	−3
2	·01	−2
1	·1	−1
0	1	0
−1	10	1
−2	100	2
−3	1000	3
$-n$	10^n	n

Hutton here assumes that Briggs suggested that the logarithm of 1 should be zero, an assumption that makes nonsense of Briggs's statement; the whole sine cannot be taken to be unity in the early part of Briggs's statement and a power of ten in the later, as would be necessary for Hutton's contention. The tablet completely misrepresents Briggs's proposal. The numbers in column B are not the logarithms of the corresponding numbers in the second column; as has been already shown, log 1 according to Briggs's proposal would be 10τ and $\log(10^n)$ would be $(10-n)\tau$ where $\tau=10^{10}$. Even if it be assumed, though I see no ground for doing so, that by unity followed by ten zeros Briggs meant unity itself, the case would be no better unless it be also assumed that the whole sine is unity.

It may be said, and indeed Hutton and many later writers seem to assume, that by 'the tenth part of the whole sine' Briggs meant 'the ratio of 1 to 10'; but when the logarithm of unity is not zero the logarithm of the ratio of 1 to 10 is not $-\log 10$ as Hutton seems to suppose. In Briggs's proposed system, exactly as in Napier's first system, the logarithm of the ratio of a to b is not $(\log a-\log b)$ but $(\log a-\log b+\log 1)$. Constant mistakes arise in the discussion of Napier's logarithms by assuming that the logarithm of a ratio is the same thing as the difference of the logarithms of its terms. Thus Hutton in describing Speidell's second table of logarithms (*Tracts*, vol. i, p. 322) says that 'Speidell's logarithm of any number n is equal to Napier's logarithm of its reciprocal $\frac{1}{n}$.' As a matter of fact, Speidell's logarithm of n is

$$\frac{1}{10}\,(\text{Napier's log } 1 - \text{Napier's log } n),$$

which is a very different number from Napier's logarithm of $\frac{1}{n}$. I do not think that Napier ever uses the phrase 'logarithm of a ratio'; he always says 'the difference of the logarithms of the terms of the ratio'—a very different thing.

Of course, if Briggs's proposal were what Hutton represents it to have been, Napier's suggestion would only have amounted to a change of sign of the logarithms, and would have been an improvement, though only a slight one. But it is exceedingly improbable that Napier would have suggested or that Briggs would have taken the trouble to record in such forcible language so slight an improvement. In the Appendix (Macdonald's *Constructio*, p. 48) Napier describes the most important improvement to be 'that which adopts a cypher as the logarithm of unity and 10,000,000,000 as the logarithm of either one tenth of unity or ten times unity' (see also p. 51). The question of sign was not, it would seem, of much importance in Napier's judgment.

No doubt, if the relation between a number and its logarithm is written in the form

$$y = ra^{\frac{x}{r}}$$

the simplification introduced by taking r to be unity and a to be 10 is very obvious; but such a method of representing the relation was just as foreign to Briggs's treatment as to Napier's, and completely conceals both the difficulty of the problem they set themselves to solve and the genius they displayed in the solution. It seems to me to be an unhappy method, when discussing the first logarithms as they came from Napier, to represent them as decimals and to speak, as is so constantly done, of the logarithm of a ratio as if it were the difference of the logarithms of its terms. It is just possible that Hutton took r, or the whole sine, to be unity without noticing the essential change that this made in the conditions; but it is, nevertheless, very hard to understand how he framed a table in which he makes Briggs propose that the logarithm of unity should be zero when in the passage under discussion Briggs so plainly stated that he meant to keep the logarithm of the whole sine zero as in the *Canon Mirificus.* The reference to the *Canon Mirificus* alone, apart from other considerations, was sufficient to show the inaccuracy of the tablet. It is very probable that Briggs saw the advantage of adapting the new logarithms 'to the round or integer numbers,' but his proposed logarithms were, in fact, the logarithms of another system than that developed in the *Arithmetica Logarithmica*; they suffered from one of the most serious defects of Napier's system.

The best commentary on Briggs's statement is to be found in the *Arithmetica Logarithmica* itself. In the first chapter Briggs defines logarithms in general to be 'numbers which are adjoined to proportional numbers and maintain equal differences' and establishes two lemmas. In the second chapter he states that it is most convenient (*commodissimum*) to use only one kind of logarithms, namely, the system in which zero is taken as the logarithm of unity, and he gives three propositions which depend upon this choice of the logarithm of unity. The first of these is that logarithms are either indices, that is, the numbers 1, 2, 3, . . ., which are adjoined to the 2nd, 3rd, 4th, . . . terms of a geometrical progression whose first term is unity, or else are proportional to such indices. The second and third propositions are the familiar theorems on the logarithm of a product and of a quotient. The use of the adjective *commodissimum* in describing the choice of the logarithm of unity may, of course, be a mere coincidence, but it certainly harmonises very well with the corresponding passage in the preface. In the third chapter he defines the system completely by fixing the logarithm, not of the whole sine but of 10, and he chooses 1,00000,00000,0000, so that his logarithms are integers of 14 figures in addition to the characteristic, which he defines in the fourth chapter. When the system is such that $\log (a^n)$ is equal to $n \log a$ it is sufficient in defining the logarithm of 10 to say, as some subsequent writers do, that 'log 10 is 1 followed by zeros,' the number of zeros depending on the number of figures in the logarithm; the number of zeros would be 7 for 7-figure logarithms. (The early logarithms were always integers.) It is interesting to notice that the radius or whole sine now disappears and, so far as dependence on Napier is concerned, Briggs's exposition is most closely connected with the various developments in the appendix to the *Constructio*, but Briggs handles the subject with the confidence of a master.

The successive stages in the evolution of the logarithm at the hands of Napier and Briggs may be stated briefly in modern notation.

Let P and P′ (Fig. 1) be the positions at time t of the points that describe the lines of the sine and the logarithm respectively; let $PZ=y$, $A'P'=x$, and $AZ=r$, the whole sine. Napier's logarithm of the sine PZ, or the number y, is x, and the connection between x and y is defined by the equations

$$\frac{1}{y}\frac{dy}{dt}=\text{constant}\ \ldots\ (1), \quad \frac{dx}{dt}=\text{constant}\ \ldots\ (2)$$

subject to the conditions $y=r$, $\frac{dy}{dt}=-V$ and $x=0$, $\frac{dx}{dt}=V$ when $t=0$. These equations give

$$y=re^{-\frac{Vt}{r}},\ x=Vt,$$

so that
$$y=re^{-\frac{x}{r}}\ \ldots,$$
and therefore

Napier's logarithm of $y=r(\log_e r-\log_e y)$ . . . . . . (N)

with $r=10^7$.

The change suggested by Briggs is most easily understood by eliminating t from equations (1) and (2), thus obtaining

$$\frac{1}{y}\frac{dy}{dx}=\text{constant}\ \ldots\ldots\ldots\ldots\ (3)$$

and integrating subject to the conditions $y=r$ when $x=0$ and $y=r/10$ when $x=10^{10}$. The integral is

$$y=r\,.\,10^{-\frac{x}{10^{10}}},$$

so that, if $r=10^{10}$,

Briggs's logarithm of $y=10^{10}(10-\log_{10} y)$ . . . . . (B).

Napier's suggestion as described by Briggs makes the integral of equation (3) satisfy the conditions $y=1$ when $x=0$ and $y=r$ when $x=10^{10}$. The integral is then

$$y=r^{\frac{x}{10^{10}}},$$

so that, if $r=10^{10}$,

Napier's suggested logarithm of $y=10^9\log_{10} y$ . . . (N′).

On a strict interpretation of the passage the logarithms would be 9-figure logarithms, but the slightest consideration of the new system would show that 10 and not the whole sine was the most suitable value of y to which a logarithm equal to a power of 10 was to be assigned.

It is of some interest to note that the account given by Hutton in the introductory pages of his *History of Logarithms* (*Tracts*, vol. i, tract 20) of the 'probable reflections' that guided the first writers on logarithms is not borne out by the treatises of these writers. Neither Napier nor Bürgi, who alone have any claim to the title of inventor of logarithms, chose zero as the logarithm of unity or any simple number as the logarithm of the ratio of 10 to 1. The conception of *ratiunculae* with which Hutton is obsessed does not seem to me to fit into Napier's ideas at all; Napier had a much more scientific conception of a logarithm than many of the subsequent writers who seem to have influenced Hutton.

In the fifth chapter of the *Arithmetica Logarithmica* Briggs states that there are two principal methods of calculating the logarithms, and that both of them are given in the appendix to Napier's *Constructio.* To what extent, if any, Napier explained these methods in his conversations with Briggs cannot now be ascertained; it is at least probable that in a visit which extended over a month the two friends would discuss the best methods of calculation, and it does not seem to me to be at all impossible that the fragmentary character of Napier's statement of the method of mean proportionals may be due to the fuller elaboration of that method in his conversations with Briggs. Dr Sang's assertion (quoted by Dr Glaisher in *Phil. Mag.* (1873), vol. 44, p. 505, note) that Napier 'carefully explained the process to be followed and delegated the actual calculation to his friend Henry Briggs of the University of Oxford' is certainly not warranted by any known facts, not even by Robert Napier's preface to the *Constructio*, but it is in the highest degree improbable that the methods of calculation were never discussed during the month that Briggs spent with Napier. It would be absurd, however, to suppose that if such discussions did take place Briggs was merely the humble pupil; the *Arithmetica Logarithmica* is an enduring monument to his eminence as a mathematician; and, as Dr Glaisher remarks, he was quite capable of devising suitable methods, and he did, in fact, develop new and ingenious simplifications. There is little doubt, I think, that Briggs valued highly the encouragement he received from Napier in carrying out his laborious calculations; a journey from London to Edinburgh was not lightly undertaken in the early years of the seventeenth century. But Napier's admiration for Briggs as a mathematician of the highest eminence was equally sincere. The friendship that existed between these two men was singularly cordial and free from petty jealousy, and it is a misfortune that either should have been represented as unfair to the other.


A
H

INTRODUCTION OF LOGARITHMS INTO TURKEY[1]

SALIH MOURAD, Lieutenant in the Turkish Navy

Ismail (Caliph Zade), as we may gather from his works, was one of the great mathematicians and astronomers who flourished in Stamboul in the eighteenth century. Unfortunately, we do not know much about his life.

We learn from rare manuscripts that his father was the chief reciter of the Koran in the Turkish army in 1746 A.D.; and he himself was a candidate for the chief recitership in 1750 A.D., as we can read at the end of one of his works, *Burhan-ul-Kifaya*. He attained the position of recitership later on, following his father's profession and serving in the Turkish army.

Young Ismail, as a clever mathematician and astronomer, had attracted the kind attention of Sultan Mustafa III when the latter was Crown Prince. He had a proficient knowledge of the French language. There are reasons to believe that he died after 1783. It was he who rendered logarithms into Turkish.

It is stated in the *Encyclopædia of Mathematics* that in the reign of Sultan Ahmed III (1713 A.D.) Mehmed Effendi, generally cited with his nickname 'Twenty-eight Chalebi,' was appointed the Turkish ambassador to France in the reign of King Louis XV.

We read in an official letter addressed to the Porte that he paid a visit to the Paris Observatory, where he met

[1] This historic note is based upon the articles:

(1) By Salih Zeki Bey in his *Encyclopædia of Mathematics* (in Turkish), the first volume published in 1896 in Stamboul.

(2) By Tahir Bey in his biographies of eminent people who lived in Smyrna.

J. Cassini, Director of the Observatory, and talked to him about the astronomical tables used in Turkey. Cassini presented him with one of the unpublished copies of the 'Tables de Astronomique' of his father, D. Cassini.

Thus the new astronomical tables were introduced into Turkey by 'Twenty-eight Chalebi,' and rendered into Turkish by Ismail Effendi by the order of Sultan Mustafa III.

The logarithms were introduced into Turkey in the early part of 1714, and rendered into Turkish in 1765 A.D. under the title of *The Translation of the Tables of Cassini.*

We read the following passages in the translator's preface :—

'The Astronomer Cassini has used the decimal system and Hindu numerals in his tables and has made the calculations by means of logarithmic tables which are not given in the text, as he took for granted that the readers were conversant with the tables. We, however, subjoin the tables as they are unknown to our readers, to whom the science of logarithms has only just been introduced.'

We can understand from these lines that the astronomical calculations were made by means of the sexagesimal system, and that logarithms were not used before the time of Ismail Effendi.

Hence Cassini's Tables replaced Olog Bey's sexagesimal system and led to the introduction of logarithms into Turkey. Indeed, Ismail's translation of Cassini's Tables is a voluminous work, consisting of a preface and an introduction, followed by fourteen chapters, concluded by a résumé of the whole. The introduction deals with the principles and application of logarithms.

The preface contains the following passage :—

'It should be known that the Franks have arranged a table under the title of logarithms which contains the logarithms of the numbers from 1 to 10,000. If it is desired to multiply two numbers, we have simply to add the logarithms of them and the sum gives the logarithm of the product. . . .

'. . . Cassini made astronomical calculations by means of tables of ratios (logarithmic tables) but did not give any idea of its principle and application in the text. The translator undertook this duty and added a résumé of its principle.'

Ismail Effendi put in the text the logarithms of numbers from 1 to 10,000, and trigonometrical tables (sine and tangent) of the angles from 0 to 45° minute by minute; the tables were arranged to the fifth decimal place.

We have it on the authority of Djevdet Pasha that the science of logarithms was invented independently in Turkey by Ismail Kelenbevi.

Djevdet Pasha, in his *History of the Ottoman Empire*, makes the following statement:—

'In the eighteenth century a Frenchman paid a visit to Constantinople and made inquiries at the Porte whether anybody in the town could understand the science of logarithms. He was taken to the house of the well-known mathematician—Ismail Kelenbevi. The poverty-stricken aspect of the house and the untidy appearance of its inmate disappointed the fashionable visitor, who could hardly expect anything learnedly advantageous from such surroundings. However, he left with Ismail Kelenbevi a book dealing with the theory of logarithms and asked him to solve or throw any light on it if he could. A few days later, when he paid a second visit, he was agreeably surprised to find that the unprepossessing Ismail had not only propounded the theory of logarithms to his entire satisfaction, but had also added something to it by working the logarithms of numbers to base 3. The Frenchman could not thereupon help exclaiming that if Ismail had been in the West, he would have been worth his weight in gold.'

This, however, is purely legendary, for Salih Zeki Bey proved that the first work on logarithms in the Turkish language was published by Ismail Effendi (Caliph Zade) in the year 1765 A.D. The author admitted therein that the science had only been introduced from the West through Cassini.

Djevdet Pasha's contention falls to the ground for the simple reason that Ismail Kelenbevi, whom he credits with the invention of logarithms, wrote his book many years after Caliph Zade.

The well-known writer, Montucla, in his first volume of *Histoire de Mathématique*, published in 1761 A.D., says:

'The Turkish government asked for a good work on Astronomy from the French Academy through Baron Tott, who was in the service of the Turkish government in the reign of Sultan Mustafa III. The French Academy sent a few works on astronomy, and according to their official note Leland's tables were included.'

The reason of the delay in the translation of Cassini's Tables by Ismail (Caliph Zade) was that there was need for a work on the logarithms which might explain the astronomical table.

One of the first copies of the Turkish translation of logarithms was bought at an auction sale by Salih Zeki Bey, the living Turkish mathematician and President of the University of Constantinople. In this book we can see that great efforts were made in the preparation of the work, especially in the careful drawing of the diagrams.

The translator of Cassini's Tables was the author of several other works, and made three dials, one of which, a horizontal dial, was placed in the court of Laleli mosque in 1762 A.D. The other two were erected on the base of the western minaret of the above-mentioned mosque. Unfortunately, these dials constructed by one of our great men have not been well looked after by later generations.

Historians of mathematics may be interested in the biography of Ismail Kelenbevi, who was credited with the invention of logarithms by Djevdet Pasha. Ismail Kelenbevi was born 1724 A.D. in Kelenbe, a village in the Smyrna district. His earliest childhood did not give promise of a bright career. He used to play with the street children, and did not care for study.

He worked hard, however, later on, and succeeded in getting a degree, equivalent to D.D. After 1758 he devoted his life to the study of science and mathematics. He studied at the house of Mehmed Effendi (Mufti Zade), who used to be playfully described as the 'walking library.' Ismail wrote one of his best works in this house.

He died in 1786, when he was professor of mathematics in the Naval College.

His works are published in Arabic, and many of them deal with science. He wrote works on arithmetic, algebra, commentary of logarithms, trigonometrical and astronomical observations.

His *Treatise of Algebra* is a good evidence of the author's complete mastery of the science. The book contains five chapters, the first four being devoted to the principles of arithmetic and proportion, and the fifth chapter treating of the determination of unknown quantities by Jebu and Mukabale (algebra and equation).

The fifth chapter is divided into different parts. He gave the old solutions for the following equations:—

(1) $ax = b$, (4) $ax + bx^2 = d$,
(2) $ax = bx^2$, (5) $d + bx^2 = ax$,
(3) $bx^2 = d$, (6) $d + ax = bx^2$,

which he called 'six questions.' In the third part of the *Algebra* he gave the solutions of thirty-five questions by means of what he called algebraic tricks. He began to write the *Algebra* in 1781, and completed it in 1783.

His *Commentary on Logarithms* is very interesting, and contains two chapters. In the first chapter he gives some idea of the principle and the invention of the logarithmic tables of numbers and trigonometrical functions; while the second chapter treats of the application of the tables. His tables also contain the logarithms of the tangents and sines of the angles from 1 to 90 degrees, thus extending the earlier tables made by Ismail Effendi (Caliph Zade).

A SHORT ACCOUNT OF THE TREATISE 'DE ARTE LOGISTICA'

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At the request of the Editorial Committee for this volume I have prepared the following abstract of the work, by John Napier of Merchiston, that bears the above title, and, after the abstract, I have arranged a few notes, for some of which I am indebted to Dr George Philip (of George Watson's College, Edinburgh), who has in the most generous way offered to me, for the purposes of this article, the fullest use of an admirable address that he read recently to the Edinburgh Mathematical Society. I take this early opportunity of expressing the obligation towards him under which his kindness has placed me.

From the nature of the treatise it seems likely that it was not quite ready for publication, although John Napier's son Robert seems to have copied it all out for the use of Henry Briggs, Professor of Geometry at Oxford; and to this fact we owe the preservation of the work; for the original manuscript was destroyed by a fire, while the transcript was preserved elsewhere and was exhibited at the Napier Tercentenary Exhibition, having been lent by its present possessor, John Spencer, of London. By the help of the Bannatyne and Maitland Clubs this document was published in 1839 as a handsome quarto volume which is not very common. It forms one of the series of volumes published by each of these Clubs, being uniform in size and binding with the other publications. The Maitland Club volumes are prefaced with the following extract from the Minutes of a meeting of the Club held in Glasgow in the hall of Hutcheson's Hospital on Saturday, the 26th of January 1839:—

'Resolved, That "The Baron of Merchistoun His Booke of Arithmeticke and Algebra" be printed for the Members, from the original manuscripts, in the possession of Mark Napier, Esquire, Advocate.'

Also in the Minutes of the Bannatyne Club it is recorded that '101 Copies, printed on Club Paper, were purchased for the members.' In most of the copies prepared for each Club an interesting list of members is given—ninety in the case of the Maitland Club and a hundred in the case of the Bannatyne Club. Other copies were also printed independently of these literary publishing Clubs. It is from the text of this work that this abstract is derived. *Note A.*

The divisions are:

Book I—26 pages, 8 chapters, 'De Computationibus.'

Book II—55 pages, 15 chapters, 'De Logistica Arithmetica.'

Book III—6 pages, 1 chapter, 'De Logistica Geometrica,' apparently incomplete.

Then follow the two books in Algebra, namely:—

Book I—25 pages, 17 chapters, 'De nominata algebrae parte.'

Book II—46 pages, 10 chapters, 'De positiva sive cossica algebrae parte' (unfinished).

To the first three books is appended a note, which the printed volume gives in facsimile: 'I could find no more of this geometricall pairt amongst all his fragments'; and the last two books end with a similar note, also in facsimile of Robert Napier's writing: 'There is no more of his algebra orderlie sett down.' From these notes it seems a fair inference that most, if not all, of the work had been transcribed, and not edited or reconstructed from 'fragments,' by Robert Napier for Mr Briggs.

The circumstances of the meeting in 1615 between the Laird of Merchiston and the English Professor of Geometry in the College founded by Sir Thomas Gresham are well known, and are described by Professor Gibson in his contribution to this volume (see page 126). Seldom, if ever, in the history of science have investigators interested in the same subject approached each other with that noble admiration and that unselfish confidence that these two men exhibited each towards the other. *Note B.*

We may now return to the consideration of the subject matter treated in each of Napier's five books. *Note C.*

Book I begins with a careful and minutely illustrated discussion of Addition and Subtraction, which he regards as fundamental. The most important point in the first chapter is the recognition of 'defective,' that is to say negative, quantities.

In the second chapter he considers multiplication and division as derived, 'ortae a primis,' from the fundamental conceptions of the preceding chapter. Multiplication presents no point of special interest, but with regard to division Napier makes the important remark that unity bears to either divisor or quotient the ratio which the other of these two bears to the dividend. Dr Philip has kindly given me the following remarks on this point:—'Using his own example we have $15 : 5 = 3 : 1$. Now by one of his laws of logarithms, which states that the logarithms of proportionals are equidifferent, we get

$$\log 15 - \log 5 = \log 3 - \log 1,$$

or,

$$\log 15 = \log 5 + \log 3 - \log 1.$$

In his first scheme Napier did not have $\log 1 = 0$, and hence the logarithm of a product is not equal to the sum of the logarithms of its factors. It is not until he changed the system and made $\log 10 = 1$ and $\log 1 = 0$ that Napier could use this.' Division is perfect when there is no remainder, imperfect when there is a remainder: and proofs of accuracy are stated (1) by remultiplication, (2) by dividing the dividend by the quotient. Chapter iii deals with calculations arising out of secondary calculations, and treats of powers and roots, a subject continued in chapter iv, in which the method for actually finding prime or composite powers is developed. Thus to form a sixth power or obtain a sixth root, we can cube and then square, take the cube root and then the square root, or *vice versa*.

The fifth chapter deals with proportion, the methods and examples being admirably set forth; and ends with a general dismissal of all kinds of miscellaneous problems which can be treated by algebra, which will be treated later, 'has ergo relinquimus, Algebram tractaturi.'

Chapter vi deals mainly with negative quantities, 'quantitantes defectivae'; their origin, their meaning, their powers, and their roots. The law of signs in multiplication and division of negative quantities is stated, by analogy, much as it was stated in the Algebras written in the middle of last century.

The double sign to even roots of positive quantities is fully explained. With regard to the even roots of negative quantities, it is pointed out that they do not exist. He reverts to this important question of imaginaries in Book III.

Chapter vii consists in a full examination of fractions: and rules for reduction, called 'abbreviatio,' and for finding the greatest common factor, are given with complete clearness. The impossibility of finding any common factor for a surd and a rational quantity is pointed out.

The eighth and last chapter of the first book gives the usual rules for adding, subtracting, multiplying, dividing, and reducing fractions, together with a note about compound fractions.

Powers and roots are dealt with, and accurate and approximate roots are given, viz.

$$\surd 16/25=\frac{4}{5},\ \surd 3/4=\surd 48/64\doteqdot\frac{7}{8}.$$

Then with one wide statement referring to the preceding rules he dismisses all the miscellaneous problems still left in dealing with fractions, and ends the book with a few remarks on improper and mixed fractions.

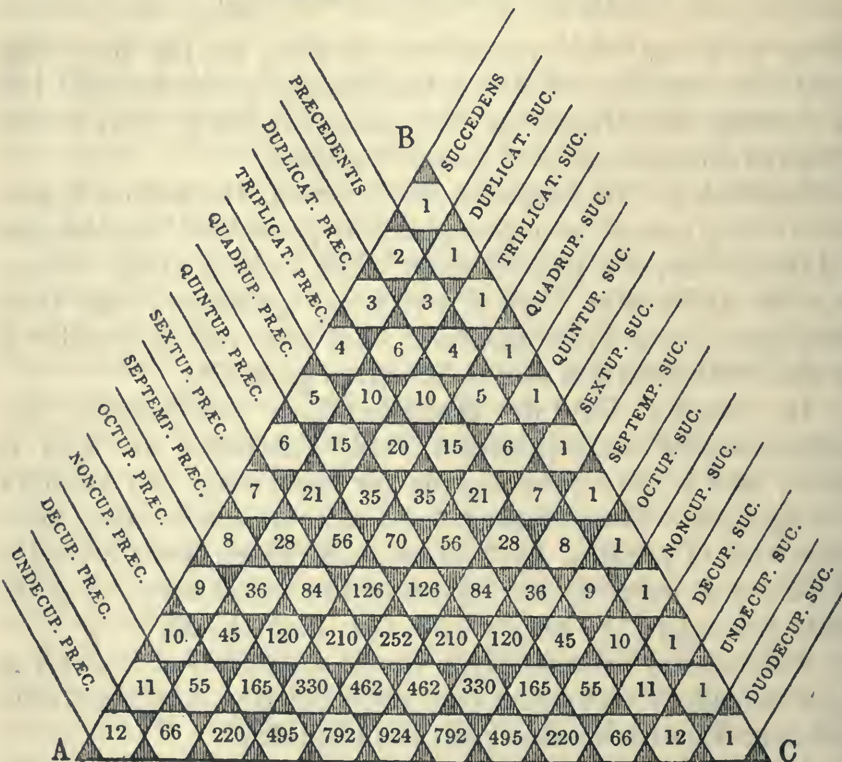
In Book II, 'De Logistica Arithmetica,' the author begins by dividing quantities into real and hypothetical 'verinomiae vel fictinomiae seu hypotheticae'—the logistic of the former he calls arithmetic, that of the latter geometry. He then discusses the decimal notation, and proceeds in chapter ii to deal with addition and subtraction generally.

In chapter iii, 'de Multiplicatione integrorum,' the common-sense multiplication table extending to 9×9 is given, and a very curious rule for recovering any product if forgotten. Thus, taking 7×8, the complementary product 3×2 is prefixed by $7-2$ or $8-3$, which gives 56—this of course is correct, but involves the recollection of 2×3. More generally $(10-a)\times(10-b)=(10-a-b)10+ab$.

The general rules, as given in the modern text-books, are then explained with great care and accuracy, and a note on compound multiplication ends the chapter.

Division is then treated, and Napier begins with $3\div5$, which he says will be considered later. Properly speaking, division is only possible when the quotient is greater than unity : and when the remainder is zero the division is perfect. Examples of short and long division follow.

Some miscellaneous rules and examples in multiplication and division occupy the next chapter. The rule for multiplication or division by 5 is given, and various extensions are given, *e.g.* 6 times=5 times+once, of which the fractional value is found in the very useful process of preparing a table of multiples for long division. The grouping of the product in his example 92105 multiplied by 865091372 is ingenious, for as soon as his one-figure product begins with a digit farther to the right than the last digit of his first product, he



writes it in the line of the first minuend, and thus attains much conciseness. The work may perhaps be given *in extenso.*

92105
865091372
736840|27631500
552630|6447350
460525|184210
828945
092105
79679240818060

The next chapter (vi) is specially interesting, treating as it does on involution and evolution. It is easier to find the fourth power of 235 by twice squaring than by repeated multiplication, and so forth for other powers. Then comes the rule involved in Euclid, ii, 4: *e.g.* $35^2=30^2+2 . 5 . 30+5^2$.

From this 352^2 is derived, and similarly $(30+5)^3$ and 351^3. Particular methods, adds Napier, can be used similarly for fourth, fifth, etc., powers, and thus tentatively for the corresponding roots, which can be found either by repeated multiplication or by repeated division, but he points out that this is not of much use, as the last quotient is rarely unity.

The same subject is continued in chapter vii, which begins virtually with the usual extension of the formula $(a+b)^2=a^2+2ab+b^2$ to the higher powers; in fact the binomial theorem is really used, and a table of coefficients given very succinctly up to those of the twelfth power. The terms $2ab+b^2$, $3a^2b+3ab^2+b^3$, he calls the 'supplement,' a the 'precedent,' b the 'succedent.'

The diagram on the opposite page is reproduced from Napier's, as given in the volume *De Arte Logistica*, p. 50. It has evidently been very carefully considered, and the geometry is fully described in the text.

The 'supplement' for the fifth power is thus obtained:—The first term is the fourth power of the precedent multiplied by the succedent multiplied by 5; the second term is the cube of the precedent multiplied by the square of the succedent multiplied by 10; and so on.

The eighth chapter applies the results of the seventh to the extraction of roots, and it is noteworthy that Napier gives an explanation of his, the usual, method that puts to shame the scanty treatment of square roots in many modern books. *Note D.*

Examples are given of 'imperfect roots,' *e.g.* $\sqrt[5]{16809} \doteqdot 7$ (2 over), and $\sqrt{164860} \doteqdot 406$ (24 over). The cube root rule is then discussed, and the repeated use of square and cube root methods for sixth, ninth, and other composite roots is duly noted. The whole chapter is an admirable piece of clear exposition.

Chapter ix shows how to amend 'imperfect roots.' For the square root of 164860 he gives a number between $406\frac{24}{813}$ and $406\frac{24}{812}$; or, in more general terms, the square root of a^2+b lies between $a+\frac{b}{2a+1}$ and $a+\frac{b}{2a}$, which is correct. His other method of emendation is to multiply the given number by a square number, and, after taking the root of this product, divide by the square root of the last-named number. His illustration is $\sqrt{50}=\sqrt{50000000}/1000$, which is greater than 7·071 and less than 7·072: Napier expresses these results as vulgar fractions.

A similar rule is given for a cube root, *i.e.* $\sqrt[3]{(a^3+b)}$ lies between $a+b/(3a^2+3a+1)$ and $a+b/(3a^2+3a)$, but this does not seem to be correct. The difference between a^3+b and $\{a+b/(3a^2+3a)\}^3$ may be either negative or positive. Take $a=2$, $b=1$, then $\sqrt[3]{9}$ ought to lie between $2+1/19$ and $2+1/18$, but it is greater than either: in fact, the upper limit should be $a+b/3a^2$ or $2+1/12$. In this connection we notice that Napier's method for approaching the irrational root of a number gives, for one of his limits, the ordinary proportional part value. We also notice that his example $\sqrt[3]{998}$ would have been better treated as $\sqrt[3]{(1000-2)}$ than as $\sqrt[3]{(729+271)}$. By an accident the root of 998 does lie between his limits in the particular case, although his general method for finding the higher limit is not correct.

Napier adds with a pleasing regard for mathematical truth: 'Hi modi, quia radices imperfectas non perficiunt, sed nimis imperfectas reddunt, Mechanicis magis quam Mathematicis placent.'

The chapter ends with a brief explanation of the use of appropriate radical signs to numbers that have no exact roots: and calls such expressions 'uninomiae or medialia,' which form the foundation of Geometrical Logistic to be hereafter treated. Meanwhile he adds it is sufficient to remember that they arise in the natural course of our study, and gives two, and in the case of the square root three, notations for expressing the roots of any number.

Not the least interesting feature of the work is the unexpected introduction, albeit in a quite natural way, of matter generally regarded as not closely related to that under consideration. Chapter x treats on the rules of the proportion of integers, but almost immediately explains what we now call the method of contracted multiplication. The rest of the chapter is upon compound proportion, and Napier then proceeds to a discussion of fractions fuller than that already given in chapter iv of this second book.

In chapters xi, xii, and xiii vulgar fractions are treated in more detail than in the earlier chapters: fractions of fractions; division and multiplication of fractions by fractions; the extraction of roots of fractions are expounded. With regard to the last, Napier states that when the root cannot be extracted exactly, we use the 'geometrical' notation of prefixing a radical sign: or we proceed mechanically, as already shown, to find two near numbers between which the root lies. This he calls the mechanical method: and he illustrates it by showing that $\sqrt{(29/4)}, = \sqrt{(290000/40000)}$, lies between 538/200 and 539/200. In the same way the mechanical cube root of 2/3 lies between 873/1000 and 874/1000.

Chapter xiv treats on the rules for proportion of fractions, and needs no comment; while the last chapter, xv, which completes these two books, and ends with the devout ascription, 'Deo autem Opt. Max. et suis Numeris omnibus Infinito, Immenso et Perfecto, retribuatur omnis laus, honor et gloria in aeternum. Amen.", defines what Napier calls 'physical fractions,' by which he means actual quantities that require for their complete expression fractions of the fundamental unit: thus an hour is a fraction of a day, and 14 hours added to 19 hours give one day and 9 hours. He refers to the cumbrous notation of astronomy, with its minutes and seconds and double figures, and compares favourably with it the obvious and simple decimal system, for which he seems at least to indicate a notation.

Book III—This book, 'De Logistica Geometrica,' opens with a somewhat elusive definition of a concrete number which seems to possess a twofold property—(1) it must refer to a real quantity, (2) it must be an incommensurable root of a number. The notation already described is repeated and extended, and the idea of an imaginary, *i.e.* actually $\surd{-9}$, is introduced with the warning that the radicle and the sign must not be transposed. The commensurability of surds, *e.g.* $\surd 12$ and $\surd 3$, is considered, and the book ends abruptly in the manner already noted. The inference is that something has been lost, for it is clear that Napier regarded his imaginaries as very important, the exact words being:

'Cumque ita radicatum uninomium sit vel abundantis vel defectivi numeri radix, ejusque index vel par vel impar—quadrifario hoc casu sequetur, quaedam uninomia esse abundantia, quaedam defectiva, quaedam et abundantia et defectiva, quae gemina dicimus; quaedam tandem nec sunt abundantia nec defectiva, quae nugacia vocamus.'

'Hujus arcani magni algebraici fundamentum superius Lib. I. cap. 6, jecimus: quod (quamvis à nemine quod sciam revelatum sit) quantum tamen emolumenti adferat huic arti, et caeteris mathematicis postea patebit.'

This passage may be rendered: 'Since then a uninomial radicate may be the root of an abundant or defective number, and its index even or odd, it follows from this fourfold cause that some uninomia are abundant, some defective, some both abundant and defective which we call double;

and some are neither abundant nor defective, which we call nugacious.

'We have already in Book I, chapter vi, above laid the foundation of this great algebraic secret: from which (although it has, as far as I know, not been revealed by any one) it will afterwards appear how great advantage will follow to this art and to the rest of mathematics.'

We are left without any further guidance as to the extent of Napier's development of imaginaries. We do know that he recognised the imaginary roots of certain quadratics, for in Book II, chapter ix, of the algebra he defines illusive equations as those to which a root cannot be found, giving as examples

$x=3x$, and $x^2=4x-5$.

It seems clear from the Latin quotation given above that Napier did develop the subject, and that he probably embodied his discoveries in a treatise now lost. But however this may be, there is no doubt that Napier's reference to imaginaries is the first on record. *Note E.*

We now proceed to the two books of algebra. *Note F.* In Book I, chapter i, he divides his subject into the algebra of nominates and that of positives. A nominate is derived from given rational or irrational numbers: the former are treated by arithmetic, the latter are the irrational roots of rational numbers.

The positive part of algebra is that which yields quantities and numbers lying concealed by fictitious suppositions; that is to say given by equations: to this part Book II is devoted, Book I dealing with nominates, of which there are three kinds—uninomia or rational numbers and their roots, plurinomia or sums and differences of uninomia, and universalia, which are all other surds apparently, although there is no specific definition given.

A distinction is drawn between even and uneven powers and roots; and the commensurability of certain radicles is considered.

Chapters ii and iii treat of the addition and subtraction of uninomia, giving certain rules when they are commensurable. Thus to add $\sqrt{12}$ and $\sqrt{3}$ we observe that $12/3=4$, of which 2 is the square root, and then adding 1, squaring, multiplying by 3, we get 27, of which the square root is the required sum. Thus

$$\sqrt{12}+\sqrt{3}=(2+1)\sqrt{3}=\sqrt{27}.$$

Similarly for subtraction.

When the surds are dissimilar, we merely get binomia or plurinomia abundant or defective as the case may be.

Chapter iv discusses the extraction of roots of uninomia: *e.g.* $\sqrt[4]{}\,.\,\sqrt{16}=\sqrt{2}$. Chapter v is interesting, although short; it deals with reduction of two surds to two similar surds: *e.g.* $\sqrt[4]{2}$ and $\sqrt[6]{5}$ are $\sqrt[12]{8}$ and $\sqrt[12]{25}$ respectively.

Chapter vi examines the multiplication and division of uninomia of any kind: and includes the powers and roots of uninomia. Chapters vii to xii attack the problems of plurinomia: chapter viii contains the important corollary that in certain cases the surd has to vanish: the word 'apotome' is introduced to represent what we call now the conjugate surd. Thus $12+\sqrt{3}$ added to $12-\sqrt{3}$ gives 24. A similar result holds in subtraction (chapter ix). In multiplication, chapter x, after a general illustration we get the rational result of multiplying an 'abundant' by a 'defective' surd: *e.g.* $\sqrt{7}+\sqrt{5}$ multiplied by $\sqrt{7}-\sqrt{5}=2$. Chapter x contains an explicit statement of the method required to naturalise a binomial surd. Thus

$$(\sqrt[3]{36}+\sqrt[3]{24}+\sqrt[3]{16})\times(\sqrt[3]{6}-\sqrt[3]{4})=6-4=2.$$

Chapter xi gives a rule for dividing by a trinomial surd; and then makes the statement that when a surd is to be divided by a dissimilar surd the result, *e.g.* $(10-\sqrt{3})\div(6+\sqrt[3]{2})$, must be left as a fraction. This is corrected by an inserted note, which refers to the chapter preceding.

The next chapter gives the now familiar rule for extracting the square root of a binomial surd, which is called a perspicuous root when it is not a more complex surd than the original, an obscure root when it is. A universal root is the root of a 'plurinomium,' and is indicated by a radical sign followed by a period, thus $\sqrt{.}$ This is understood to apply to all that follows: *e.g.*

$$\sqrt{.}\,10-\sqrt{2}\equiv\sqrt{(10-\sqrt{2})}.$$

Chapters xiii to xvii discuss the treatment of 'universals' without arriving at any result that involves much more than new notations for certain processes; and with a general synopsis of results the first Book of the Algebra ends.

The second and concluding Book of the Algebra is longer than any of the preceding, and extends, although uncompleted, to 46 pages of print. The book begins with the discussion of integral powers, and their multiplication and division. The ordinary index laws are expounded with a curious notation which can be illustrated best by taking one of his examples slightly modernised. If we wish the cube root of $O^{30}a^{12}$, where O is a mere symbol for any number and a for any other number, we have $O^{10}a^4$; similarly the square root is $O^{15}a^6$; the sixth root O^5a^2; and 'there is no other root,' adds Napier. He proceeds to cases with numerical coefficients, *e.g.* $64x^6$, and finds the same roots. The notation is rather puzzling, and it would appear that Napier uses O, a, ℞, indiscriminately, as we use x, y, z, . . . and he speaks of extracting a root where indices only are expressed, much as we might say the cube root of ${}^3a^6b^9$ is ${}^1a^2b^3$, where any quantity may be attached to the first index. This seems to involve a suggestion of symbolism: and is of course correct if properly interpreted.

The multiplication of different powers of the same quantity is discussed, and the ordinary errors are pointed out in chapter v.

In chapter vi the arrangement of what we should now call a polynomial, where $\sqrt{x}$ is the argument in place of x, is explained and illustrated; any missing terms being completed by the introduction of a zero coefficient. The next chapter deals with division of polynomials by polynomials: the method is that generally in use now, but the arrangement is a good deal longer. Chapter viii investigates in a complete and logical manner the extraction of various roots of polynomials, distinguishing the cases when the roots are respectively exact and inexact: among the examples given are $\sqrt[3]{(x^3-10x^2+31x-30)} \doteq x-10/3$, remainder $-7x/3+190/27$ and $\sqrt{(x^2+4ax+a^2-4bx-4ab+4b^2+4x+4a-8b-61)}$, which is $x+a-2b+2$ with remainder $2ax-65$.

The rest of the chapter is interesting by reason of the attempt, under an assumed equation between x, a, etc., to approach more nearly to a root by proper treatment of the remainder. The diction is rather involved, and the new terms 'formal,' 'informal,' are used of remainders respectively without and with a positive term: 'formable' and 'informable,' 'reformable,' 'reformation,' 'reformatrix' are other new terms. Briefly, a 'reformatrix' is an identity between the algebraic symbols which, in substitution, leads to the 'reformation' of the root and the remainder, and to the ultimate attainment of a remainder which is either zero, or formal, or formable, *i.e.* less informal than the original one.

Although, to me at least, the passages are not perfectly clear, and there are certain misprints in the text, the general idea seems to be the attainment of an approximate root not only algebraically but arithmetically—a result only possible when some relation or relations exist between the symbols.

An elementary analogy may be found by dividing $2x+3$ by $x+1$, the result being of different form according to the values assumed by x.

The remaining chapters of the unfinished book are given to equations. Chapter ix defines certain terms, and incidentally refers to imaginary roots of quadratics in § 10. Chapter x first discusses the transposition, etc., of terms, and then proceeds to the other methods of simplification, *e.g.* $\sqrt{(2x+5)}=\sqrt{(3x-4)}$ leads to $x=9$; $\sqrt[3]{(2x-6)}=3x$ leads to $27x^3-2x+6=0$. Perhaps the removal of 'radices universales' is one of the most important parts of the chapter. One of the cases taken is:

$$\sqrt{(3x-2)}+\sqrt{(2x+1)}=\sqrt{(4x+2)},$$

whence

$$5x-1+2\sqrt{(6x^2-x-2)}=4x+2,$$

and

$$4(6x^2-x-2)=(3-x)^2.$$

In the text there are several misprints.

Napier, without giving any general method, reduces the surd equation $12-\sqrt{x}=x$ to the form $x^2-25x+144=0$, and states that although the latter has two roots, 9 and 16, yet only one of them (9) satisfies the former. He adds, 'ut postea patebit'; but there is no suggestion of any further explanation in the three remaining articles, which end abruptly with the addendum by Robert Napier, already quoted.

NOTES AND COMMENTS

Note A—Mark Napier's excellent introduction to the Latin text of the *De Arte Logistica*, pp. i-xciv, is worthy of careful study: the sequence of composition of this work and of the *Canon Mirificus* is clearly made out, the latter succeeding the former, after an interval of twenty years, in 1614. John Napier's death occurred in 1617.

Robert Napier in 1619 published his father's posthumous work, and gave generous yet exact credit to Henry Briggs, 'who,' he says, 'undertook most willingly the very severe labour of this Canon' (*i.e.* the Briggsian tables) 'in consequence of the singular affection that existed between him and my father of illustrious memory,—the method and explanation of its use being left to the inventor himself. But now, since he has been called from this life, the whole burden of the business rests upon the learned Briggs, as if it were his peculiar destiny to adorn this Sparta.'

Note B—Nor was Briggs alone in paying due homage to Napier's inventiveness. The immortal Kepler, when he had read the *Descriptio Canonis Mirifici*, addressed to the author a warm-hearted and enthusiastic letter of congratulation, which reached Scotland after Napier's death.

Note C—It may be convenient to gather together some of the dates bearing on the progress of arithmetical and algebraic printing.

Lucas de Burgo printed a volume in 1494.

Cardan printed the next known book on the subject in 1539; he died in 1575.

Michael Stifellius published at Nuremberg his *Arithmetica Integra* in 1544; and introduced the signs +, −, and $\surd$, derived from *r*, to denote the root (radix) of a number.

Robert Recorde published the first work in English, 1552; he added the sign = to the symbols in use.

Simon Stevinus of Bruges published *La Practique d'Arithmetique* about 1582; and suggested the idea of decimal subdivision of the unit.

Vieta, in France, was a contemporary of Napier's, and extended the theory of equations. It is probable that Napier never saw his works, which were first collected into one volume by Schooten in 1646.

Note D—It is interesting to notice that although Napier invented an excellent notation of his own for expressing roots, he did not make use of it in his algebra, but retained the cumbrous and in some cases ambiguous notation generally used in his day. His notation was derived from this figure

1	2	3
4	5	6
7	8	9

in the following way :—|_| prefixed to a number means its square root, ‾|_| its fourth root, |‾|_| its fifth root, |‾ its ninth root, and so on, with extensions of an obvious kind for higher roots.

Note E—'Historians of algebra usually credit Girard with being the first to use imaginary roots of equations, but in view of the above the Flemish mathematician must waive his claim in favour of Napier. As Girard's most important work was published in 1629, there is no question of Napier having got the idea from him, and it is superfluous to remark that Girard could not have borrowed from Napier' (Dr Philip).

Note F—'The value of Napier's work will be better understood if we remember that in his day and for some time afterwards mathematics did not include algebra and arithmetic, which were represented merely by a collection of rules without any attempt at logical connection. Geometry and mathematics were practically synonymous terms. Present-day detractors of Greek mathematics should keep in mind that until a comparatively modern date it formed the only rigorously deductive part of our subject, and saved mathematics from being a mere collection of disconnected rules' (Dr Philip).

THE FIRST NAPIERIAN LOGARITHM CALCULATED BEFORE NAPIER

Giovanni Vacca, Professore incarito of Chinese in the Royal University of Rome

In the ordinary histories of mathematics there are very few suggestions about the way in which John Napier conceived the idea of his great discovery, truly one of the most beautiful made by man, not only as supplying a new method for saving time and trouble in tedious calculations, but also as forming one of the most important steps towards the discovery of the infinitesimal calculus.

Generally the only reference made is to the ψαμμίτης of Archimedes.[1]

I have lately observed that in the *Summa de Arithmetica* of Fra Luca Paciolo, printed in Venice in 1494, there is the following problem :

(Fol. 181, n. 44.) ‘A voler sapere ogni quantità a tanto per 100 l’anno, in quanti anni sarà tornata doppia tra utile e capitale, tieni per regola 72, a mente, il quale sempre partirai per l’interesse, e quello che ne viene, in tanti anni sarà raddoppiato. Esempio: Quando l’interesse è a 6 per 100 l’anno, dico che si parta 72 per 6; ne vien 12, e in 12 anni sarà raddoppiato il capitale.’

Luca Paciolo says that the number of years necessary to double a capital placed at compound interest, is the number resulting from the division of the fixed number 72 by the rate of interest per 100.

If we try to explain the mystery of this number 72 (and

[1] Cf. J. B. Biot, *Journal des savants*, 1835 ; aud also G. Loria, *Le scienze esatte nell' antica Grecia*, pp. 757, 970.

the reason of this mystery was impenetrable to the succeeding arithmeticians, for instance, Tartaglia), we easily see in modern notation that

$$\left(1+\frac{r}{100}\right)^x=2$$

or, taking Napierian logarithms :

$$x\log\left(1+\frac{r}{100}\right)=\log 2$$

and to a first approximation, if r is small :

$$x=\frac{100\log 2}{r}$$

therefore 72 is only a rough calculation of the number 100 log 2.

This problem is to be found, without explanation, in modern treatises, for instance in the Introduction to the *Tables d'intérêt composé* of Pereyre.

Sometimes the number 70 is given instead of 72.

If this problem were known to Napier, might it not have been a suggestion leading to his further discovery ? Perhaps a research in his manuscripts can explain this point.

In any case it is curious to note that the Napierian logarithm of 2 was printed before the year 1500, with an approximation of 3 per 100.

THE THEORY OF NAPIERIAN LOGARITHMS EXPLAINED BY PIETRO MENGOLI (1659)

GIOVANNI VACCA, Professore incarito of Chinese in the Royal University of Rome

The Italian mathematician, Pietro Mengoli, the last disciple of Bonaventura Cavalieri in Bologna, has not yet received the just appreciation he merits.

I have vindicated his discoveries in two lectures read in the Universities of Genoa and Rome, in 1910 and 1911; and recently, in an accurate article in his *Bibliotheca Mathematica*, G. Eneström has deduced the same conclusion from the analysis of the works of Mengoli.

The *Geometria Speciosa* of Pietro Mengoli, published in 1659 in Bologna, contains an elementary, purely arithmetical, and rigorous theory of Napierian logarithms.

This theory has also the great advantage of being *direct*, that is, does not depend on the complicated theory of powers of numbers with irrational exponents.

The theory of Mengoli is explained in pages 69-75 of his *Geometria Speciosa.* But his style and notation are difficult. I shall therefore expound it in modern language, with the hope that it may be introduced into elementary mathematics.

Given the integer number n, greater than 1, let us consider the two successions:

$$\begin{cases} 1+\frac{1}{2}+\dots+\frac{1}{n-1},\ \frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n-1},\ \frac{1}{3}+\dots+\frac{1}{3n-1},\ \dots \\ \frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n},\ \frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{2n},\ \frac{1}{4}+\dots+\frac{1}{3n},\ \dots \end{cases}$$

The first succession is decrescent; for if, from the p^{th} term,

$$\frac{1}{p}+\frac{1}{p+1}+\dots\frac{1}{pn-1}$$

we subtract the next in order,

$$\frac{1}{p+1}+\dots+\frac{1}{(p+1)n-1},$$

we obtain $\frac{1}{p}-\left(\frac{1}{pn}+\frac{1}{pn+1}+\dots+\frac{1}{(p+1)n-1}\right),$

and this difference is positive $\Big($because each of the n fractions in parenthesis is either $\frac{1}{pn}$ or less than $\frac{1}{pn}\Big)$.

The second succession is crescent, for a similar reason. Every term of the first succession is greater than the corresponding term of the second; the difference between the corresponding terms of order p is

$$\left(\frac{1}{p}+\dots+\frac{1}{pn-1}\right)-\left(\frac{1}{p+1}+\dots+\frac{1}{pn}\right)=\frac{1}{p}-\frac{1}{pn}=\frac{1}{p}\left(1-\frac{1}{n}\right);$$

taking p sufficiently great, we may make this difference as small as we please.

Hence every term of the first succession is greater than every term of the second, and the two successions define a single real number. This number, depending on the integer n, may be called $\log n$.

For every positive integer value of p, we have:

$$\frac{1}{p}+\frac{1}{p+1}+\dots+\frac{1}{pn-1}>\log n>\frac{1}{p+1}+\dots+\frac{1}{pn}\dots(1).$$

It is easy now to prove the fundamental property of logarithms, namely, that if m, n are two integer numbers, greater than 1, we have

$$\log m+\log n=\log(mn).$$

Thus, if p is a positive integer number, we have from (1):

$$\frac{1}{p}+\ldots+\frac{1}{mp-1}>\log m>\frac{1}{p+1}+\ldots+\frac{1}{mp},$$

$$\frac{1}{mp}+\ldots+\frac{1}{mnp-1}>\log n>\frac{1}{mp+1}+\ldots+\frac{1}{mnp}.$$

Adding together these inequalities:

$$\frac{1}{p}+\ldots\frac{1}{mnp-1}>\log m+\log n>\frac{1}{p+1}+\ldots+\frac{1}{mnp}.$$

But from (1), for every value of p, we have also $\log (mn)$ contained between the same quantities, therefore:

$$\log (mn)=\log m+\log n$$

Q.E.D.

This theory may also be presented in a geometrical form, and generalised to fractional values; but this is not necessary if we want only to understand the construction and the meaning of a table of Napierian logarithms of integer numbers.


A H

NAPIER'S RULES AND TRIGONOMETRICALLY EQUIVALENT POLYGONS

D. M. Y. Sommerville, M.A., D.Sc., F.R.S.E., Lecturer in Mathematics, University of St. Andrews[1]

§ 1. Napier's Rules for a right-angled spherical triangle were published in the *Mirifici logarithmorum canonis descriptio*, Lib. II, cap. iv. They are often expressed in textbooks on spherical trigonometry as if they were mere mnemonics, and have been thus regarded by men like Airy and De Morgan, who, one would have expected, might have appreciated their proper setting.

They are expressed by Napier in logarithmic form, which shows that, as in most of his mathematical work, he looked from the point of view of the computer. But this does not necessarily imply that Napier himself considered them as mere useful rules. In fact, he calls them *theorems*. And, while he verifies them in the ordinary way by testing each of the known relations between the parts of a right-angled spherical triangle, he exhibits their true character in relation to the star-pentagon with five right angles.

§ 2. If any two of the five parts of a right-angled spherical triangle are given, the remaining parts are determined; hence there is a relation connecting every set of three parts. The ten relations which exist are of different forms, for there is a distinction in kind between the various parts of the triangle, the hypotenuse, the two sides, and the two angles. Napier's procedure reduces these relations to uniformity by replacing the triangle, with its three kinds of parts, by a pentagon in

[1] Now Professor of Mathematics, Wellington University, New Zealand.

which each of the five sides bears the same relation to the whole figure.

This pentagon is obtained from the spherical triangle A′CD (Fig. 1), with hypotenuse c, sides a and b, and opposite angles α and β, by retaining the sides but replacing the two vertices by their polars. The resulting intersecting or star-pentagon A′C′E′B′D′ has its sides in order, $\pi-c$, $\pi-\alpha$, $\frac{\pi}{2}+b$, $\frac{\pi}{2}+a$, $\pi-\beta$. (In the figure $\bar{c}$ stands for $\frac{\pi}{2}-c$.) At the same time a simple pentagon

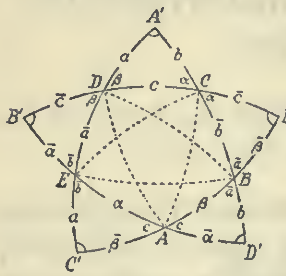


Fig. 1

ABCDE is produced whose sides are c, $\frac{\pi}{2}-a$, α, β, $\frac{\pi}{2}-b$, and whose opposite vertices are the poles of the sides—*i.e.* it is a self-polar pentagon; its angles are the supplements, $\pi-c$, $\frac{\pi}{2}+a$, $\pi-\alpha$, $\pi-\beta$, $\frac{\pi}{2}+b$, of the opposite sides, and are thus the same as the sides of the rectangular pentagon. Again, the polar of the rectangular star-pentagon is a quadrantal star-pentagon ACEBD whose angles are equal to the sides of the self-polar pentagon.

§ 3. Now in any of these pentagons the five sides or the five angles, whichever set are not quadrantal, form a homogeneous set of five parts, which form only two kinds of triads. Namely, if the parts, in cyclic order, are denoted by a, b, c, d, e, the parts of a triad are either *consecutive*, as bcd, or *separated*, as ace. In the first case the extremes, b and d, are adjacent to the mean c, but are separated from one another; in the second case the extremes, a and e, are separated from the mean c, but are adjacent to each other.

There are therefore only two forms of relations among triads, say

$$c=f(b,\ d) \text{ for a consecutive triad,}$$
$$c=\mathrm{F}(a,\ e) \text{ for a separated triad.}$$

Both f and F are symmetrical functions of the two variables. The two functions f and F are connected; for, substituting $b=f(a,\ c)$ and $d=f(c,\ e)$ in the first equation, we have

$$c=f\{f(a,\ c),\ f(c,\ e)\},$$

and this relation, which connects c, a, e, must be identical with

$$c=\mathrm{F}(a,\ e).$$

Hence if f is known, F is determined.

§ 4. The given conditions limit the form of the function, without, however, completely determining it. To illustrate the way in which its form is restricted we shall confine ourselves to a special form which leads to the actual equations of spherical trigonometry.

Suppose

$$f(c)=bd \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(A)}$$

to be a possible form of the triadic relationship connecting the cyclically arranged numbers a, b, c, d, e. Then if ϕ is any function whatever, we may replace the given numbers by $\phi(a)$, $\phi(b)$, . . ., so that

$$f\{\phi(c)\}=\phi(b)\phi(d)$$

is also a possible form of the triadic relationship.

Thus (A) includes the more general form

$$\psi(c)=\phi(b)\phi(d).$$

§ 5. Consider therefore the triadic relation (A). If a and b are given, the other three numbers are determined, viz.

from $f(b)=ac$ $\qquad c=\dfrac{f(b)}{a}$,

from $f(a)=be$ $\qquad e=\dfrac{f(a)}{b}$,

and $\qquad f(d)=ce=\dfrac{f(a)f(b)}{ab}$.

The last relation is the triadic relation connecting a separated triad. By taking the numbers in the order *acebd* and renaming them a, b, c, d, e, we find that

$$f(c)=\frac{f(b)}{b}\cdot\frac{f(d)}{d}$$

is another possible form of the triadic relationship; or, putting b for $\frac{f(b)}{b}$, etc.,

$$\phi(c)=bd,$$

where, if $y=\phi(x)$, $x=\frac{f(y)}{y}$.

§ 6. Taking the other two relations between triads,

$$f(c)=bd \text{ and } f(e)=da,$$

substitute the values of c, d, e in terms of a and b and we get

$$\frac{f(a)f(b)}{ab}=f(d)=f\left\{\frac{1}{b}f\left(\frac{f(b)}{a}\right)\right\}$$
$$=f\left\{\frac{1}{a}f\left(\frac{f(a)}{b}\right)\right\}$$

which must be identities.

§ 7. Suppose $f(x)$ can be expanded in ascending powers of x:

$$f(x)\equiv u_0+u_1x+u_2x^2+\ldots$$

Then we have $(u_0+u_1a+u_2a^2+\ldots)(u_0+u_1b+u_2b^2+\ldots)$

$$\equiv ab\left[u_0+\frac{u_1}{b}f\left(\frac{f(b)}{a}\right)+\frac{u_2}{b^2}\left\{f\left(\frac{f(b)}{a}\right)\right\}^2+\ldots\right]$$

$$\equiv ab\left[u_0+\frac{u_1}{b}\left\{u_0+\frac{u_1}{a}(u_0+u_1b+\ldots)+\frac{u_2}{a^2}(u_0+u_1b+\ldots)^2+\ldots\right\}\right.$$
$$\left.+\frac{u_2}{b^2}\left\{u_0+\frac{u_1}{a}(u_0+u_1b+\ldots)+\ldots\right\}^2+\ldots\right].$$

The right-hand side cannot be integral in a and b unless u_2, u_3, . . . all vanish. We may assume therefore

$$f(x)=u_0+u_1x.$$

Then $(u_0+u_1a)(u_0+u_1b)\equiv ab\left[u_0+\frac{u_1}{b}\left\{u_0+\frac{u_1}{a}(u_0+u_1b)\right\}\right]$

$$\equiv u_0ab+u_0u_1a+u_1^3b+u_0u_1^2.$$

Therefore $u_0^2=u_0u_1^2,\ u_0u_1=u_1^3,\ u_1^2=u_0.$

These are all satisfied by $u_0=u_1^2$.

Then $f(x)=u_1^2+u_1x=u_1(u_1+x),$

and the triadic relationship is

$$u_1(u_1+c)=bd$$

or

$$1+\frac{c}{u_1}=\frac{b}{u_1}\cdot\frac{d}{u_1}.$$

Hence, replacing a, b, c, d, e by u_1a, etc., we have as the simplest form of the relationship [1]

$$1+c=bd,$$

and the general relationship is

$$1+\phi(c)=\phi(b)\phi(d)$$

where ϕ is any function.

§ 8. Thus, if $\phi(x)=-\sin^2 x$, we have $\cos^2 c=\sin^2 b\sin^2 d$

or $$\pm\cos c=\sin b\sin d.$$

If $\phi(x)=\tan^2 x$, $\sec^2 c=\tan^2 b\tan^2 d$

or $$\pm\cos c=\cot b\cot d.$$

The triadic relationship for a separated triad corresponding to the relationship $1+c=bd$ for a consecutive triad is

$$1+c=\frac{(1+a)(1+e)}{ae}=\left(1+\frac{1}{a}\right)\left(1+\frac{1}{e}\right),$$

and that corresponding to $1+\phi(c)=\phi(b)\phi(d)$ is

$$1+\phi(c)=\left(1+\frac{1}{\phi(a)}\right)\left(1+\frac{1}{\phi(e)}\right).$$

For $\phi(x)=-\sin^2 x$

$$1+\frac{1}{\phi(x)}=1-\operatorname{cosec}^2 x=-\cot^2 x,$$

and the relationship is

$$\pm\cos c=\cot a\cot e.$$

Similar forms can be obtained with hyperbolic instead of circular functions.

[1] This form of the relationship was given by Gauss, 'Pentagramma mirificum,' *Werke*, iii, p. 484.

§ 9. Corresponding to (1), the *self-polar pentagon* ABCDE whose sides a, b, c, d, e in order are connected by the relations

$$\cos a = \sin b \sin e = \cot c \cot d,$$

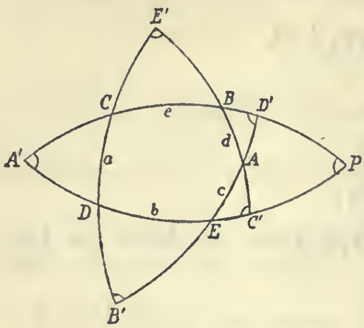


Fig. 2

we have a number of other polygons with five parts connected by relations of similar form. We have, namely,

(2) The *intersecting rectangular pentagon* A′C′E′B′D′ whose sides are $\pi-a$, $\pi-c$, $\pi-e$, $\pi-b$, $\pi-d$.

(3) The *right-angled triangle* A′CD with hypotenuse a, sides $\frac{\pi}{2}-b$ and $\frac{\pi}{2}-e$, and opposite angles c and d.

(4) The *trirectangular quadrilateral* AC′PD′ with angle $\pi-a$, and sides $\frac{\pi}{2}-c$, e, b, $\frac{\pi}{2}-d$.

These are figures in elliptic geometry which are trigonometrically equivalent to the right-angled triangle.

§ 10. In hyperbolic geometry there is a similar series of equivalent polygons.

Starting with the *right-angled triangle* APQ, let one vertex become ideal. Then the lines PQ and AQ are non-intersecting, and have a common perpendicular BC. The figure PABC is then a *trirectangular quadrilateral*. Let P also become ideal, so that QP and AP have a common perpendicular DE. Then ABCDE is a *rectangular pentagon*.

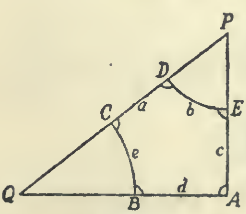


Fig. 3

Let the sides of the pentagon be a, b, c, d, e.

Then for the trirectangular quadrilateral the angle is ib, and the sides are $c+i\frac{\pi}{2}$, d, e, $a+i\frac{\pi}{2}$, which are possibly indeterminate to a multiple of the period $i\pi$.

For the right-angled triangle the hypotenuse is $a+i\pi$, the sides $c+i\frac{\pi}{2}$ and $d+i\frac{\pi}{2}$, and the opposite angles ie and ib, with similar indeterminateness. These are connected by relations of the form

$$\cosh(a+i\pi)=\cosh\left(c+i\frac{\pi}{2}\right)\cosh\left(d+i\frac{\pi}{2}\right)=\cot ib\cot ie.$$

These reduce to

$$\cosh a=\sinh c\sinh d=\coth b\coth e,$$

which are the relations connecting the sides of the rectangular pentagon.

§ 11. These relations form the subject of a paper by A. Ranum.[1] He treats also of the polygons which are equivalent to the general triangle.

The triangle ABC being given, A′B′C′ are the absolute poles of the sides. The two triangles ABC, A′B′C′ are in perspective with their common orthocentre as centre and their common orthaxis as axis of perspective. They intersect to form a hexagon $P_2P_3Q_3Q_1R_1R_2$ with all its angles right. The sides of this rectangular hexagon are connected by the same relations as the sides and angles of the triangles; viz. if a, b', c, a', b, c' are the sides in order, then

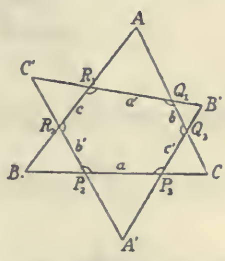


Fig. 4

$$\left.\begin{aligned}\frac{\sinh a}{\sinh a'}=\frac{\sinh b}{\sinh b'}=\frac{\sinh c}{\sinh c'}\\ \cosh a=-\cosh b\cosh c+\sinh b\sinh c\cosh a'\\ \cosh b\cosh c'=\coth a\sinh b+\coth a'\sinh c'.\end{aligned}\right\}$$

These are the fifteen tetradic relations connecting every set of four parts of the hexagon.

By replacing one or more of the sides of the hexagon by angles, we may obtain a *quadrirectangular pentagon* or

[1] 'Lobachefskian polygons trigonometrically equivalent to the triangle' (*Jahresber. D. Math.-Ver.*, 21 (1913), 228-240).

a *birectangular quadrilateral* as polygons trigonometrically equivalent to the triangle.

§ 12. Similarly in elliptic geometry we find a *rectangular hexagon* (intersecting) $P_2P_3Q_3Q_1R_1R_2$ equivalent to the triangle ABC; a *quadrirectangular pentagon*, with concave angle, $AQ_3P_3P_2R_2$; a *birectangular quadrilateral* $AR_2A'Q_3$ with right angles opposite, or AR_2P_2C with right angles adjacent.

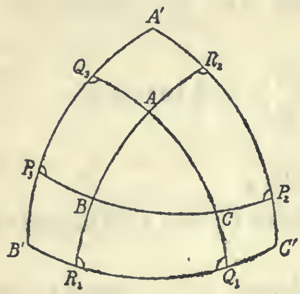


Fig. 5

If a, b', c, a', b, c' are the sides of the rectangular hexagon, the relations connecting them are

$$\left.\begin{aligned} \frac{\sin a}{\sin a'} = \frac{\sin b}{\sin b'} = \frac{\sin c}{\sin c'} \\ -\cos a = \cos b \cos c + \sin b \sin c \cos a' \\ \cos b \cos c' = \cot a \sin b + \cot a' \sin c'. \end{aligned}\right\}$$

BIBLIOGRAPHY OF BOOKS EXHIBITED AT THE NAPIER TERCENTENARY CELEBRATION, JULY 1914

R. A. SAMPSON, M.A., D.Sc., F.R.S., F.R.S.E., Astronomer Royal for Scotland, Professor of Astronomy, University of Edinburgh

The books described below may be classified in the following divisions :—

1. Napier's work on the Apocalypse, in its various editions.
2. The editions of the Description and Construction of Logarithms.
3. The editions of the Rabdology.
4. *De Arte Logistica.*
5. The calculations of Briggs, Gunter, Vlacq, Kepler, and Ursinus.
6. References to the co-discovery of logarithms by Jobst Buergi.
7. The *Opus Palatinum* of Rheticus, with the additions of Pitiscus—the great table of natural sines, etc., preceding Napier's discovery.
8. References to the method of *Prosthaphœresis,* an earlier alternative for facilitating multiplications.
9. Specimens illustrating the subsequent history of logarithmic tables.

In preparing this collection much use has been made of Dr J. W. L. Glaisher's admirable article on 'Logarithms' in the *Encyclopœdia Britannica,* and in the bibliographical descriptions of W. R. Macdonald's catalogue appended to his English version of the *Constructio,* published in 1889.

The descriptions of Napier's works are somewhat condensed from those of Macdonald—to which reference should be made for full details of pagination, etc.—but the books have been compared with his description and any variations noted.

The Society is indebted for the loan of these volumes to John Spencer, Esq., F.I.A.; Archibald Scott Napier, Esq.; L. Evans, Esq.; W. R. Macdonald, Esq.; Dr Hay Fleming; J. Ritchie Findlay, Esq.; Prof. H. Andoyer; University College, London; the Universities of Edinburgh and Glasgow; the Royal Observatory, Edinburgh (Crawford Library); and the Town Library, Dantzig. For the sake of completeness a few books are described of which it was not possible to obtain copies for exhibition; the titles of these are enclosed in [ ].

I.—NAPIER'S WORK ON THE APOCALYPSE

1. A Plaine Dis- | couery of the whole Reue- | lation of Saint Iohn: set | downe in two treatises: The | one searching and prouing the | true interpretation thereof: The o- | ther applying the same paraphrasti- | cally and Historically to the text. | Set foorth by | Iohn Napeir L. of | Marchistoun younger. | Whereunto are | annexed certaine Oracles | of Sibylla, agreeing with | the Reuelation and other places | of Scripture. | Edinbvrgh | Printed by Ro- | bert Walde-graue, prin- | ter to the Kings Ma- | jestie. 1593. | Cum Priuilegio Regali.

At the four corners the words: Pax, Amor, Infesta malis, Pacis alumnus.

The title-page and the coat of arms of James VI and Anne of Denmark[1] are reproduced in Plates VII and VIII.

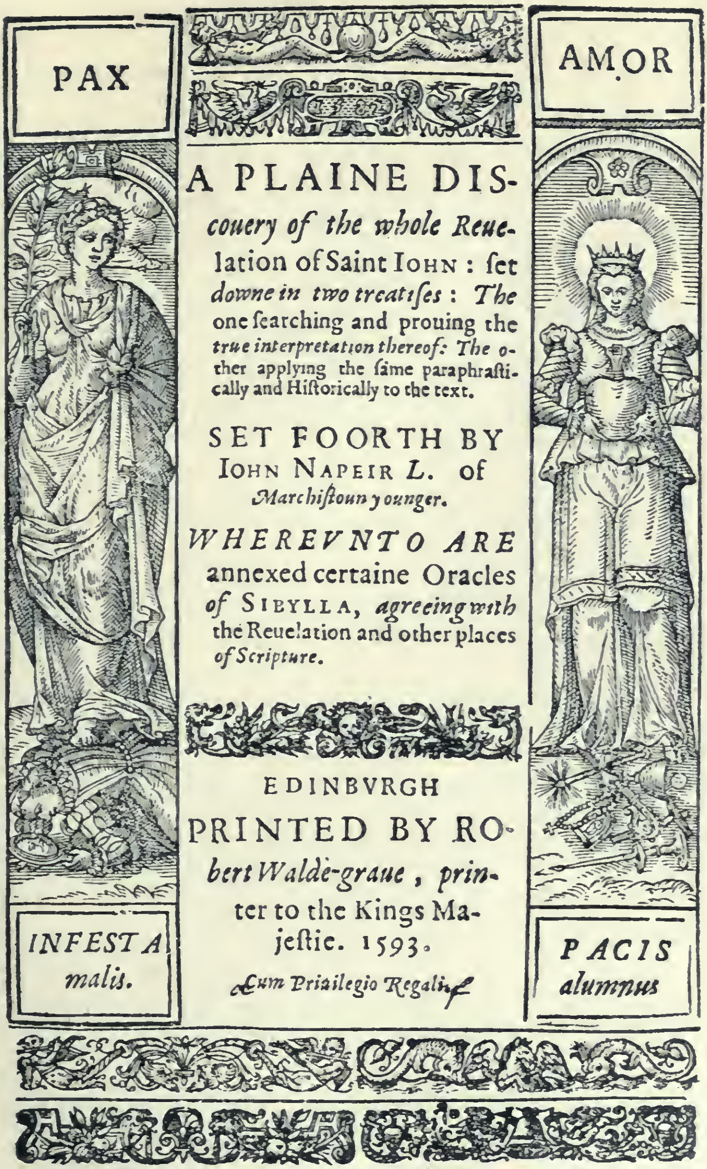
Lent by W. Rae Macdonald, Esq., F.F.A.

[1] Mr W. Rae Macdonald gives the following account of the coat of arms:—

A shield bearing impaled arms:

Dexter for Scotland.—A lion rampant within a double tressure flory counter-flory, the latter dimidiated.

Sinister for Denmark.—Quarterly. 1st, Three lions passant guardant in pale. 2nd, A lion rampant crowned holding with his fore paws a battle-axe with curved



PAX

AMOR

A PLAINE DIS-
couery of the whole Reue-
lation of Saint IOHN : ſet
downe in two treatiſes : The
one ſearching and prouing the
true interpretation thereof: The o-
ther applying the ſame paraphraſti-
cally and Hiſtorically to the text.

SET FOORTH BY
IOHN NAPEIR *L.* of
Marchiſtoun younger.

WHEREVNTO ARE
annexed certaine Oracles
of SIBYLLA, *agreeing with*
the Reuelation and other places
of Scripture.

EDINBVRGH
PRINTED BY RO-
bert Walde-graue , prin-
ter to the Kings Ma-
ieſtie. 1593.

Cum Priuilegio Regali.

INFESTA
malis.

PACIS
alumnus

IN VAINE ARE AL EARTHLIE CONIVNCTIONS, VNLES VVE BE HEIRES TOGETHER, AND OF ONE BODIE, AND FELLOVV PARTAKERS OF THE PROMISES OF GOD IN CHRIST, BY THE EVANGELL.

2. Reprints of the above.

London, Printed for John Norton. 1594.

Lent by Dr Hay Fleming.

[Edinburgh, Printed by Andrew Hart. 1611.]
[London, Printed for John Norton. 1611.]
[Edinburgh, Printed for Andro Wilson, 1645.]

3. Ovvertvre | De Tovs Les | Secrets De | l'Apocalypse | ov Revelation | De S. Iean. | . . . | Par Iean Napeir (c.a.d.) Nonpareil | Sieur de Merchiston, reueue par lui-mesme | Et mise en François par Georges Thomson Escossois. | . . .

A La Rochelle. | Par Iean Brenovzet, demeurant pres | la boucherie Neufue. | 1602.

4° ; size $6 \times 8\frac{1}{2}$ inches ; A-Z, Aa-Hh.

Lent by Archibald Scott Napier, Esq.

4. Ovvertvre | De Tovs Les Secrets | De | L'Apocalypse | Ov Revelation | De S. Iean. | . . . | Par Iean Napeir (c.à.d. Nonpareil) Sieur de | Merchiston . . . | Et mise en François par Georges | Thomson Escossois. | Edition Troisieme. |

haft (Norway). 3rd, Three open crowns (Sweden). 4th, Semé of hearts a lion passant—the 1st and 4th quarters are probably both intended for Denmark, viz. Or semé of hearts gules three lions passant guardant in pale azure. Dividing the quarters the cross of Dannebrog, viz. A cross gules charged with an another argent. In a champagne in base a wyvern passant wings elevated (Vandalia).

En surtout.—Quarterly. 1st, Two lions passant guardant in pale (Schleswig). 2nd, An escutcheon between three demi nettle leaves and as many passion nails in pairle (Holstein). 3rd, A swan gorged with a crown (Stormarn). 4th, A mounted knight armed at all points (Ditmarchen).

Sur le tout du tout.—Parted per pale. *Dexter,* A cross pattée alisée (Delmenhorst). *Sinister,* Two bars (Oldenburg).

Surrounding the lower part of the shield the Collar of the Thistle with Jewell of St. Andrew.

Above the shield a helmet front face, with mantling, ensigned with an imperial crown and thereon for crest a lion sejant crowned holding in his dexter paw a sword and in his sinister a sceptre.

Motto, on an escroll issuing from behind the crest : IN MY DEFENC—GOD ME DEFEND.

Supporters.—Dexter, A unicorn royally gorged holding a banner of Scotland. Sinister, a wyvern holding a banner charged with the same.

A La Rochelle, | Par Noel De La Croix. | CIↃ. IↃC. VII.

8°; size $4\frac{1}{2} \times 5\frac{1}{4}$ inches; A-Z, Aa-Og.

Lent by ARCHIBALD SCOTT NAPIER, Esq.

5. Een duydelijcke verclaringhe | Van de gantse Open- | baringhe Ioannis des Apostels. | . . . Wt-ghegheven by Iohan Napeir, Heere | van Marchistoun, de tonghe. | . . . | Overgheset . . . Door | M. Panneel. . . .

Middelburch, | By Symon Moulert, Boeck-vercooper, woonen- | de op den Dam, inde Druckerije. Anno 1607. |

8°; size $4\frac{1}{2} \times 6\frac{1}{2}$ inches; A-Z, Aa-Dd.

Lent by ARCHIBALD SCOTT NAPIER, Esq.

6. Iohannis Napeiri, | Herren zu Merchiston, | Eines trefflichen Schottländischen | Theologi, schön und lang gewünscht | Auslegung der | Offenbarung Jo- | hannis, | . . .

Getruckt zu Franckfort am | Mayn, Im Jahr 1615. |

8°; size 4×7 inches; A-Z, Aa-Nn.

Lent by ARCHIBALD SCOTT NAPIER, Esq.

7. Napiers | Narration: | Or, | An Epitome | Of | His Booke On The | Revelation. | . . .

London. | Printed by R. O. and G. D. for Giles Calvert. 1641. |

4°; size $5 \times 6\frac{3}{4}$ inches; A-C.

Lent by ARCHIBALD SCOTT NAPIER, Esq.

II.—THE DESCRIPTION AND CONSTRUCTION OF THE CANON OF LOGARITHMS

8. Mirifici | Logarithmorum | Canonis descriptio, | Ejusque usus, in utraque | Trigonometria; ut etiam in | omni Logistica Mathematica, | Amplissimi, Facillimi, & | expeditissimi explicatio. | Authore ac Inventore, | Ioanne Nepero, | Barone Merchistonii, | &c. Scoto. | Edinburgi, | Ex officinâ Andreæ Hart | Bibliopôlæ, CIↃ. DC. XIV.

MIRIFICI

Logarithmorum

Canonis deſcriptio,

Ejuſque uſus, in utraque *Trigonometria; ut etiam in* omni Logiſtica Mathematica, *Ampliſsimi, Facillimi, &* *expeditiſsimi explicatio.*

Authore ac Inventore, IOANNE NEPERO, Barone Merchiſtonii, *&c. Scoto.*

EDINBURGI,
Ex officinâ ANDREÆ HART
Bibliopôlæ, CIↃ. DC. XIV.

The title-page of the copy in the Royal Observatory Library is reproduced on Plate IX.

4°; size $7\frac{1}{2} \times 5\frac{1}{2}$ inches.

The volume contains: A, Title, Dedication to Charles, Prince of Wales, laudatory verses. B-D2, *Mirifici . . . Canonis . . . explicatio*, liber i, containing the ideas of arithmetical and geometrical progressions described concurrently at the same rate, the definition of a logarithm (of a sine) in the words 'Logarithmus ergo cujusque sinus, est numerus quàm proxime definiens lineam, quae aequaliter crevit interea dum sinus totius linea proportionaliter in sinum illum decrevit, existente utroque motu synchrono, atque initio aequiveloce.' The particular choice of initial equal velocity and increase of the logarithm for decrease of the number are in effect a choice of base, namely e^{-1}. The term *antilogarithm* is used for what is now called the logarithm of the cosine, and *differential* for the logarithm of the tangent. The book continues with a demonstration of the properties of logarithms in respect to proportional numbers, the extraction of square and cube roots. Pp. 14, 15 are incorrectly numbered 22, 23. D3-I: liber ii, 'De canonis mirifici logarithmorum præclaro usu in Trigonometria.' Contains logarithmic methods for solution of plane triangles, the 'Rules for Circular Parts' for right-angled and quadrantal spherical triangles, with their demonstration, and applications to the logarithmic solution of general spherical triangles, followed by (I2-m2) the table of logarithms to single minutes in the following form:—

Gr. 30 + | −

30 min.	*Sinus.*	*Logarithmi.*	Differentiæ.	logarithmi.	Sinus.	
0	5000000	6931469	5493059	1438410	8660254	60
					......	
					59	

There is here no explicit use of decimal fractions. The sine was a line of so many units, and the logarithm was the like. It appears that Napier did use decimal fractions in nearly the modern form for the construction of the logarithms. See in this connection the *Construction* and the *Rabdology*, Nos. 10, 16 below. It will be noted that in modern notation $\cdot 6931469 = \log_e 2 = \log_{1/e} \cdot 5$.

The last page (m2) is blank.

Lent by the ROYAL OBSERVATORY.

9. The same as (8), except that the last page (m2) contains the following *Admonitio*, expressing an intention of publishing later an improved form of logarithm, which may have referred to a system to base 10, as put more explicitly in Wright's translation, 1616 (No. 13 *q.v.*), and in the dedication of the Rabdology :—

ADMONITIO

Quum hujus Tabulae calculus, qui plurimorum Logistarum ope et diligentia perfici debuisset, unius tantùm opera et industria absolutus sit, non mirum est si plurimi errores in eam irrepserint. Hisce igitur sive à Logistae lassitudine, sive Typographi incuria profectis ignoscant, obsecro, benevoli Lectores : me enim tum infirma valetudo, tum rerum graviorum cura praepedivit, quo minus secundam his curam adhiberem. Verùm si huius inventi usum eruditis gratum fore intellexero, dabo fortasse brevì (Deo aspirante) rationem ac methodum aut hunc canonem emendandi, aut emendationem de novo condendi, ut ita plurium Logistarum diligentia, limatior tandem et accuratior, quàm unius opera fieri potuit, in lucem prodeat.

Nihil in ortu perfectum.

Lent by the UNIVERSITY OF EDINBURGH.

10. Mirifici | Logarithmo- | rvm Canonis | Descriptio, | Ejusque usus, in utraque Trigonome- | tria ; vt etiam in omni Logistica Ma- | thematica, amplissimi facillimi, | & expeditissimi explicatio. |

MIRIFICI

LOGARITHMO-
RVM CANONIS
DESCRIPTIO,

Ejuſque uſus, in utraque Trigonometria; *vt etiam in omni Logiſtica Mathematica, ampliſsimi, facillimi,* *& expeditiſſimi explicatio.*

ACCESSERVNT OPERA POSTHVMA;

Primò, Mirifici ipſius canonis conſtructio, & Logarithmorum ad naturales ipſorum numeros habitudines.

Secundò, Appendix de alia, eáque præſtantiore Logarithmorum ſpecie conſtruenda.

Tertiò, Propoſitiones quædam eminentiſsimæ, ad Triangula ſphærica mirâ facilitate reſolvenda.

Autore ac Inventore IOANNE NEPERO, Barone Merchiſtonii, &c. Scoto.

EDINBVRGI,
EXCVDEBAT ANDREAS HART.
ANNO 1619.

Accesservnt Opera Posthvma; | Primò, Mirifici ipsius canonis constructio, & Logarith- | morum ad naturales ipsorum numeros habitudines. | Secundò, Appendix de alia, eáque præstantiore Loga- | rithmorum specie construenda. | Tertiò, Propositiones quædam eminentissimæ, ad Trian- | gula sphæ-rica mirâ facilitate resolvenda. | Autore ac Inventore Ioanne Nepero, | Barone Merchistonii, &c. Scoto. | Edinbvrgi, | Ex-cvdebat Andreas Hart. | Anno 1619.

4°; size $7\frac{3}{4}\times 6$ inches.

The title-page, which is reproduced in Plate x, was printed so as to suit the binding of the Description and Construction in one volume. In all cases, however, whether the *Descriptio* is bound in with it or not, the *Constructio* has its own special title-page without ornamental border. After this (second) title there follows the preface of Robert Napier to the reader (A1, 2),—from which we learn that Napier called his numbers 'artificial numbers,' and had this treatise written out some years before the word logarithm was invented,—and: 'Mirifici Logarithmorum Canonis Constructio; (Qui et Tabula Artificialis ab autore deinceps appellatur) eorumque ad naturales ipsorum numeros habitudines' ($A3^1$-$E4^1$). The method of the Construction is in essence the following:—the values of $(1-{\cdot}0000001)^m$ for $m=1, 2, \ldots 100$ are readily formed and lie between 1 and $1-{\cdot}00001$; those of $(1-{\cdot}00001)^n$ for $n=1, 2, \ldots 50$ lie between 1 and $1-{\cdot}0005$; those of $(1-{\cdot}0005)^p$ for $p=1, 2, \ldots 21$ lie between 1 and $1-{\cdot}01$; those of $(1-{\cdot}01)^r$ for $r=1, 2, \ldots 69$ lie between 1 and $\cdot 5$; hence in the product of four members out of these four sequences we have the means of expressing any number between 1 and $\cdot 5$, and if a value be adopted for $\log(1-{\cdot}0000001)$, we have the means of finding the logarithm of any number. The first stage was to form 69 tables, each of 21 columns corresponding to the values of $(1-{\cdot}01)^r(1-{\cdot}0005)^p$, and the other sequences were used for interpolating between numbers in these tables. The value adopted for log 9999999 was 1·0000001. This makes Napier's logarithm of N equal to $\log_e(e/N)$, or to the complement of $\log_e N$, taking N as less than unity.

In calculating $(1-{\cdot}00001)^{50}$, which in Napier's notation is 9995001,224804, a small error occurred, and Napier gives 9995001,222927 as the result. This error affects the logarithms of 1—·0005, 1—·01, and thence the whole table, and by accumulation renders incorrect the last digit of Napier's notation as published in the *Descriptio*.

In an Appendix, $E4^2$-$F3^1$, pp. 40-42, Napier speaks of a better system of logarithms, in which the logarithm of unity should be zero, while the unit should be the logarithm of ten or of one-tenth. He gives also the outline of a second method of constructing logarithms by finding proportionate numbers for interpolating by repeated extractions of square roots, the chief method employed by Briggs in his *Arithmetica Logarithmica*, and a third method by consideration, as we should put it, of the equation $N^p=10^{p\log_{10}N}$, the number of digits on the left, divided by p, giving an approximate value of $\log_{10} N$. In particular, if p is a power of 10 we obtain an upper and a lower limit for the logarithm, differing by a unit in the last place. Illustrations of these methods are found in Briggs's *Arithmetica Logarithmica*, in Macdonald's translation of the *Constructio*, and elsewhere.

$F4^2$-$I2^1$ contains propositions on the solution of spherical triangles, without auxiliary right-angled or quadrantal triangles; one of the celebrated propositions known as Napier's Analogies,

$$\tan\tfrac{1}{2}(b-c)=\tan\tfrac{1}{2}a\sin\tfrac{1}{2}(B-C)/\sin\tfrac{1}{2}(B+C),$$

is at the end, the other three being added in Briggs's notes, which follow.

A commentary by Briggs ($F3^2$-$G3^1$) upon this Appendix follows.

Lent by the Royal Observatory.

11. Logarithmorum | Canonis Descriptio, | Sev | Arithmeticarvm Svppvtationvm | Mirabilis Abbreviatio. | Eiusque vsus in vtraque Trigonometria vt etiam in omni | Logistica Mathematica amplissimi, facillimi & | expeditissimi explicatio. | Authore ac Inuentore Ioanne Nepero, | Barone Merchistonii, &c. Scoto. |

Lugduni, | Apud Barth. Vincentium. | M. DC. XIX. | Cum Privilegio Caesar: Majest. & Christ. Galliarum Regis. |

In the same volume:

Mirifici | Logarithmorvm | Canonis Con- | strvtio; | Et Eorvm Ad Natvrales | ipsorum numeros habitudines; | Vna Cvm Appendice, De Alia | eâque praestantiore Logarithmorum specie condenda. | Quibus accessere Propositiones ad Triangula sphae- | rica faciliore calculo resoluenda: | Vnàcum Annotationibus aliquot doctissimi D. Henrici | Briggii in eas et memoratam appendicem. | Authore & Inuentore Ioanne Nepero, Barone | Merchistonii, &c., Scoto. |

Lvgdvni, | Apud Bartholomæum Vincentium, | Sub Signo Victoriæ. | M. DC. XX. | Cum priuilegio Caesar. Maiest. & Christ. Galliarum Regis. |

4°; size $5\frac{1}{2}\times7\frac{1}{4}$ inches; A-H3, A-M, and A-H_3.

This is the Lyons Edition, being a reprint of Napier's two works. At the end of the tables, on $H3^2$, is *Extraict du Priuilege du Roy . . .* and *Achevé d'imprimer le premier Octobre mil six cents dix-neuf.*

Lent by ARCHIBALD SCOTT NAPIER, Esq.

12. Logarithmorum | Canonis Descriptio | . . .
Lvgdvni, | Apud Barth. Vincentium. | M. DC. XX.

Mirifici | Logarithmorum | Canonis Con- | strvtio; | . . .
Lvgdvni, | Apud Bartholomaeum Vincentium, | sub Signo Victoriae. | M. DC. XX. | . . .

This is identical with the foregoing except for the date upon the title-page of the *Descriptio*, and the addition at the end of the *Constructio* of *Extraict du Priuilege du Roy*

. . . and *Mirifici Logarithmorum*, Achevé d'imprimer le 31. Mars 1620.

Lent by Archibald Scott Napier, Esq.

13. A | Description | Of The Admirable | Table of Loga- | rithmes, | With | A Declaration Of | The Most Plentiful, Easy, | and speedy vse thereof in both kindes | of Trigonometrie, as also in all | Mathematicall calculations. | Invented and Pvbli- | shed In Latin By That | Honorable L. Iohn Nepair, Ba- | ron of Marchiston, and translated into | English by the late learned and | famous Mathematician | Edward Wright, | With an Addition of an Instrumentall Table | to finde the part proportionall, inuented by | the Translator, and described in the end | of the Booke by Henry Brigs | Geometry-reader at Gresham- | house in London. | All perused and approued by the Author, & pub- | lished since the death of the Translator. | London. | Printed by Nicholas Okes. | 1616.

The title-page and first two chapters of this book are reproduced in Plates I to VI, p. 32.

12°; size $5\frac{3}{4}\times3\frac{1}{4}$ inches.

Contains: Title, Dedication by Samuel Wright, the son of Edward Wright, to the East India Company, Napier's Dedication to Prince Charles, Preface to the Reader by Henry Briggs, Napier's preface to this translation, commendatory verses, tabular view of the book, and table of errata (A1-12). The words 'and maintain' at the bottom of A5[1] have not been ruled out in this copy, as they have been in all copies described by Macdonald, nor is the page A10 cut out, with its curious verses. These verses are perhaps worth transcribing.

To the worthily honored Author and Translator.

Pull off your Laurel rayes, you learned Greekes,
Let Archimed and Euclid both give way,
For though your pithie sawes have past the pikes
Of all opponents, what they e'er could say,
And put all modern writers to a stay.

Yet were they intricate and of small use,
Till others their ambiguous knots did loose.

And bonnets vaile (you Germans) Rheticus,
Reignoldus, Oswald, and Iohn Regiomont,
Lansbergius, Finckius, and Copernicus,
And thou Pitiscus, from whose cleerer font
We sucked have the sweet from Hellespont:
For were your labours ne're compos'd so well,
Great Napier's worth they could not paralell.

By thee great Lord, we salve a tedious toyle,
In resolution of our trinall lines,
We need not now to carke, to care, or moile,
Sith from thy witty braine such splendor shines,
As dazels much the eyes of deepe Diuines.
Great thy inuention, greater is the praise,
Which thou unto thy Nation hence dost raise.

Nor are we less oblig'd to thee good Wright,
By whose industrious paines are vulgar made
Not onely those, but things that in the night
Haue ear'st lyen hid, and couered in the shade,
Till thou into the Ocean seas didst wade,
And there foule errors didst discouer more,
Than any witty Mariner before.

And though thy fatall threed be cut in twaine,
Thy pilgrimage come to a common end,
Yet as a patterne do thy workes remaine,
To such braue spirits as do still intend
Their deerest lives in sacred Arts to spend,
As Heriots, Gunter, Briggs and Torperley
Who to the world rare secrets can display,
But such is now th' ingratitude of time,
That hees the wisest which doth lowest clime.

THOMAS BRETNOR, Mathem.

The translation runs from $B1^1$ to $E9^1$, and is stated in Napier's preface to be 'precisely conformable to my minde and the originall.' It contains, however, a significant addition —namely, at the foot of p. 19:

An Admonition

But because the addition and subtraction of these former numbers may seeme somewhat painfull, I intend (if it shall please God) in a second Edition, to set out such Logarithmes as shal make those numbers aboue written to fall upon decimal numbers, such as 100,000,000, 200,000,000, 300,000,000, etc. Which are easie to bee added or abated to or from any other number.

The table follows ($E9^2$-$I6^1$), taken to one place less than the original, except from 89° to 90°, for which the full number of places is given, the last digit being marked by a point, being the earliest instance of the use of the simple decimal notation. Thus we have:

Deg. 0. + | −

Min.	Sines.	Logarith.	Differen.	Logarith.	Sines.	
30	8726	4741385	4741347	38·1	999961·9	30
					......	Min.

Deg. 89.

A folding plate follows, being a triangle with transversals, for reading proportional parts graphically, devised by Wright with an explanation of its use by Briggs.

Lent by the Royal Observatory.

14. A | Description | Of The Admirable | Table of Loga- | rithmes: With | A Declaration of the most Plenti- | full, Easie, and Speedy vse there- | of in both kinds of Trigonome- | try, as also in all Ma- | thematicall calcu- | lations. | Inuented and published in Latine by that | Honourable Lord Iohn Nepair, Baron of | Marchiston, | and translated into Eng- | lish by the late learned and famous | Mathematician, Edward | Wright. | With an addition of the Instrumentall Table | to find the part Proportionall, intended | by the Translator, and described in the end of the | Booke by Henrie Brigs Geometry- | reader at

Gresham-house in | London. | All perused and approued by the Authour, and | published since the death of the Translator. | Whereunto is added new Rules for the | ease of the Student. |
London, | Printed for Simon Waterson. | 1618. |

The book is the same as the foregoing, with a slight change and addition to the title, and corresponding to it, following Briggs's account of proportional parts, eight pages containing 'An Appendix to the Logarithmes, shewing the practise of the Calculation of Triangles, and also a new and ready way for the exact finding out of such lines and Logarithmes as are not precisely to be found in the Canons.'

Lent by the UNIVERSITY OF EDINBURGH.

15. The Construction | of the Wonderful Canon of | Logarithms | By | John Napier | Baron of Merchiston | Translated From Latin Into English With Notes | And | A Catalogue | Of The Various Editions Of Napier's Works, By | William Rae Macdonald, F.F.A. |
William Blackwood And Sons | Edinburgh And London | MDCCCLXXXIX. |

4°; size 8×10 inches.

This is one of the most important works on Napier. The catalogue, with full bibliographical detail, is made with great care and fulness, and contains references to many public libraries at which the different editions may be found.

Lent by the ROYAL OBSERVATORY.

III.—THE EDITIONS OF THE RABDOLOGY

16. Rabdologiæ, | Sev Nvmerationis | Per Virgulas | Libri Dvo: | Cum Appendice de expeditis- | simo Mvltiplicationis | Promptvario. | Quibus accessit & Arithmeticæ | Localis Liber vnvs. | Authore & Inventore Ioanne | Nepero, Barone Mer- | chistonii, &c. | Scoto. |
Edinbvrgi, | Excudebat Andreas Hart, 1617. |

The title-page of this is reproduced in facsimile in Plate XI.

12°; size $5\frac{5}{8}\times 3\frac{1}{4}$ inches.

¶1¹-6², Title, Dedication to Lord Dunfermline, laudatory verses, Table of Contents, and the two lines:

Quae terrêre solent ab amore Matheseōs, illa
Hoc parvo invenies esse remota libro.

A1¹-B9² (pp. 1-42), 'Rabdologiae liber primus de usu Virgularum numeratricium in genere.' B10¹-D9² (pp. 43-90), 'Rabdologiæ liber secvndvs de usu Virgularum Numeratricium in Geometricis and Mechanicis officio Tabularum.' D10¹-E8² (pp. 91-112), 'De expeditissimo multiplicationis promptuario Appendix.' E9¹-G5² (pp. 113-154), 'Arithmeticæ Localis, quæ in Scacchiæ abaco exercetur, Liber unus.'

Book I contains the method of calculating by rods; it is in essence a mechanical multiplication table up to nine times, with any multiplicand. If, for example, there be written upon three rods the multiples of 2, 6, 7, separating the tens and units by an oblique line, and the rods be placed side by side, we should have, for the product by 8, the digits $|{}^1/_6|{}^4/_8|{}^5/_6|$. The units of each rod are the tens of the next following rod upon the right; uniting these we read the product, $267\times 8=6+13\times 10+10\times 100+1000=2136$. Plates showing the complete set of ten rods are given on pp. 6-8. The method is a very practical one, and probably saved many errors as well as the tedium attached to heavy multiplications; but Glaisher's remark may be quoted, that 'nothing shows more clearly the rude state of arithmetical knowledge at the beginning of the seventeenth century than the universal satisfaction with which [this] invention was welcomed by all classes.' On pp. 21, 22 are some remarks upon decimal arithmetic, and two notations are given, a simple point or comma and a set of accents, thus:

$$1993,273 \text{ or } 1993,2'\,7''\,3'''.$$

PLATE XI

RABDOLOGIÆ, SEV NVMERATIONIS PER VIRGULAS

Ex LIBRI DVO: Libris

Cum APPENDICE de expeditissimo MVLTIPLICATIONIS PROMPTVARIO.

Quibus accessit & ARITHMETICÆ LOCALIS LIBER VNVS.

Authore & Inventore IOANNE NEPERO, *Barone* MERCHISTONII, &c. SCOTO.

Biblioth: Edinensis

EDINBVRGI,
Excudebat *Andreas Hart*, 1617.

The book treats further of the extraction of square and cube roots. The rules are expressed in mnemonic doggerel.

Book II gives tables showing the relations of sides and areas for regular figures of given dimensions, and the sizes of circles inscribed and circumscribed to these, with the like for regular solids; and a table of the relative weights and capacities of vessels made of stone and of certain metals.

The method of the Promptuary, which follows, is a more complicated system of engraved rods or strips, for mechanical multiplication, than that described in Book I. The concluding work, Local Arithmetic, is, in modern terms, the expression of any number in the scale of radix 2, in which consequently every digit is either 0 or 1, and a mechanical method of multiplication based upon this, by means of an areal abacus or enlarged chess board, the multiplier being represented by points upon one side and the multiplicand by points upon another. The method is applied further to the extraction of square roots.

Lent by the ROYAL OBSERVATORY.

17. Rabdologiae | Sev Nvmerationis | per Virgulas libri duo: | Cum Appendice de expe- | dissimo Multiplicationis | Promptvario. | Quibus accessit & Arithme- | ticae Localis Liber unus. | Authore & Inventore Ioanne | Nepero, Barone Merchisto- | nij, &c. Scoto. |
Lvdvni. | Typis Petri Rammasenij. | M. DC. XXVI.

12°; size $3\frac{1}{8} \times 5\frac{3}{8}$ inches; †, A-G4.

The Leyden Edition. On p. 39 Napier's notation of accents, $0' \, 2'' \, 5'''$, for decimal fractions is replaced by the clumsy notation of Stevinus, ⓪, 2 ①, 5 ②, though the remarks and notation on p. 21 are given as in the Edinburgh Edition.

Lent by ARCHIBALD SCOTT NAPIER, Esq.

18. Raddologia, | Ouero | Arimmetica Virgolare | In due libri diuisa; | con appresso un' espeditissimo | Prontvario Della

Molteplicatione, | & poi un libro di | Arimmetica Locale : | Quella mirabilmente commoda, anzi vtilissima | à chi, che tratti numeri alti ; Questa curiosa, & diletteuole à chi, che sia d'illustre ingegno. | Auttore, & Inuentore | Il Baron Giovanni Nepero, | Tradottore dalla Latina nella Toscana lingua | Il Cavalier Marco Locatello, | Accresciute dal medesimo alcune consi- | derationi gioeueuoli. |

In Verona, Appresso Angelo Tamo, 1623. | Con licenza de' superiori.

8°; size $6\frac{1}{8} \times 4\frac{1}{8}$ inches.

† 1-8, Title, Translator's Dedication to Teodoro Trivultio, verses, Table of Contents, and Imprimatur. $A1^1$-$F4^1$ (pp. 1-95), 'Della Raddologia Libro Primo.' $F5^1$-$K4^1$ (pp. 97-159), 'Della Raddologia Libro Secondo.' $K5^1$-$N3^2$ (pp. 161-210), 'Prontvario.' $N4^1$-$Q8^1$ (pp. 211-269), 'Arimmetica Locale.' $Q8^2$, Colophon.

As Macdonald remarks, the translation is not exact, and at the end Napier is made to ascribe all the glory and honour 'alla Beatissima Vergine Maria.'

Lent by the ROYAL OBSERVATORY.

19. Rhabdologia | Neperiana | Das ist | Newe vnd sehr leichte | art durck etliche Stäbchen allerhand Zah- | len ohne mühe vnd hergegen gar gewiss zu Multiplici- | ren vnd zu dividiren, auch die Regulam Detri, vnd beyderley ins | gemein vbliche Radices zu extrahirn : ohne allen brauch | des sonsten vb-vnnd nützlichen | Einmahl Eins | alss in dem man sich leichtlich | verstossen kan | Erstlich erfunden durch einen vornehmen Schottlän- | dischen Freyherrn Herrn Johannem Neperum | Herrn zu Merchiston, &c. | Anjtzo aber auffs kürtzeste alss jmmer müglich gewesen | nach vorhergehenden gnugsamen Probstücken | ins Deutsche vbergesetzt | Durch | M. BENJAMINEM URSINUM, Churf. Bran- | denburgischen Mathematicum | cum gratia et privilegio. |

Gedruckt zum Berlin im Grawen Kloster durch George Rungen | Im Yahre Christi 1623.

4°; 24 pages; size $7\frac{1}{2} \times 6\frac{5}{8}$ inches.

First Book only. The diagrams of the 'Bones' are shown on four plates at the end.

Lent by LEWIS EVANS, Esq.

A second copy, without the plates, was lent by the University College, London.

IV

20. Manuscript of *De Arte Logistica.* This was in the possession of the Napier family up till a recent date, and has the following inscription by the seventh Lord Napier within the cover:

John Napier of Merchistoune, Inventor of the Logarithms, left his Manuscripts to his Son *Robert,* who appears to have caused the following Pages to have been written out fair, from his Father's Notes, for Mr. Briggs, Professor of Geometry, at Oxford. They were given to Francis, the 5th Lord Napier, by William Napier of Culcreuch, Esq[r.], Heir Male of the above named *Robert.* Finding them in a neglected state amongst my family Papers, I have bound them together in order to preserve them entire.

1801. (Signed) NAPIER.

On the front is written

The Baron of Merchiston his booke of Arithmeticke and Algebra.

with some other words defaced; also down the same page

For Mr Henrie Briggs professor of Geometrie at Oxforde.

and on the back

Merchistouns Algebra.

Including these two pages the MS. includes 140 pages, size 6×8 inches; the whole, according to Mark Napier, in the hand of Robert Napier. The manuscript was transcribed and printed in 1889 by Mark Napier (see No. 21 below) who gave it the title *De Arte Logistica* from its opening words 'Logistica est ars bene computandi.'

It has a modern binding of red morocco, lettered *Ioannis Napier Fragmenta,* and within as a bookplate the arms of Lord Napier.

Lent by J. SPENCER, Esq., F.I.A.

21. De Arte Logistica | Joannis Naperi | Merchistonii Baronis | Libri Qui Supersunt. |
Impressum Edinburgi | M. DCCC. XXXIX.

4°; size $8\frac{1}{4}\times10\frac{1}{2}$ inches.

This was published by Mark Napier, from the manuscript (20) above, then in the possession of Lord Napier. It bears as a second title: 'The Baron of Merchiston His Booke of Arithmeticke and Algebra. For Mr Henrie Briggs Professor of Geometrie at Oxforde.' The book is evidently an early one, as the notation for decimal fractions, which he employed for logarithms, is not used The Algebra deals with surds and their composition. Perhaps the most interesting part is the clear treatment, on p. 86, of imaginary quantity. See Professor Steggall's detailed account in this volume, pp. 146-161.

Lent by the ROYAL OBSERVATORY.

V.—THE CALCULATIONS OF BRIGGS, GUNTER, VLACQ, KEPLER, AND URSINUS

[22. Logarithmorum | Chilias Prima, | Quam autor typis excudendam curavit, non eo con- | cilio ut publici juris fieret, sed partim, ut quorun- | dam suorum necessariorum desiderio privatim satis- |

faceret: partim, ut eius adiumento, non solum chilia- | des aliquot insequentes; sed etiam integrum Loga- | rithmorum canonem, omnium Triangulorum cal- | culo inservientem commodius absolueret . . . | Quod autem hi Logarithmi, diversi sint ab ijs, | quos Clarissimus inventor, memoriae semper colendae, | in suo edidit Canone Mirifico; sperandum eius librũ | posthumum, abunde nobis propediem satisfactu- | rum Qui autori (cum eum domi suae, Edinburgi, bis inviseret, et apud eum humanissime exceptus, | per aliquot septimanas libentissime mansisset; eique horum partem praecipuam quam tum absoluerat | ostendisset) suadere non destitit, ut hunc in | se laborem susciperet. Cui ille non | invitus morem gessit.

In tenui; sed non tenuis, fructusve laborve.]

16 pp.

This small pamphlet (British Museum, 717 C. 11 (1)) has the name 'Henry Briggs' written on the title-page or preface quoted above. It is undoubtedly the work referred to in a letter from Sir Henry Bourchier to Usher, quoted in the *Lives of the Gresham Professors*, of date December 6, 1617: 'Our kind friend, Mr Briggs, hath lately published a supplement to the most excellent tables of logarithms, which I presume he has sent to you.' The reference to Napier's death shows that it cannot have been earlier than 1617. It is the earliest table to base 10. The page runs:

	Logarithmi.			Logarithmi 9.			
467 8	2	6693, 16880, 56612	501 2	2	6998, 37725, 86726	8660	
	(34 lines)						
500	2	6989, 70004, 33602	4	2	7275, 41257, 02856		

The difference columns on the right were originally blank, but some of the differences, to four figures, are written in, and the line separating the characteristic from the mantissa is ruled in by hand.

[23. Canon | Triangulorum, | sive | Tabulae Sinuum et Tangen- | tium artificialium ad Radium | 10000,0000, & ad scru- | pula prima quadrantis. | Per Edm. Gunter, Professorem | Astronomiae in Collegio | Greshamensi. | Londini | Excudebat Gulielmus Jones. | MDCXX.]

8º; A-F.

Contains: Dedication to John Egerton, Earl of Bridgewater, Description, and Tables of Logarithmic Sines and Tangents, to seven places, in the form:

M.	Sin. 30.				Tan. 30.				
0	9698	9700	9937	5306	9761	4393	10238	5606	60

This is the earliest table of logarithmic sines, etc., to base 10.

In the British Museum, 717 C. 11 (2).

24. Canon | Triangvlorum, | Or Tables of | Artificall Sines and Tangents, to | to a Radius of 10000,0000 parts, and | each minute of the Quadrant. | By Edm. Gunter, Professor of Astronomie in | Gresham Colledge. |
London, | Printed by William Iones, for Iames Bowles, and are to be | sold at the Marigold in Pauls Church-yard. | 1636.

This is a reprint, in Gunter's *Works* (2nd edition), of (23) above, in larger format but with practically the same arrangement of the page.

Lent by J. Ritchie Findlay, Esq.

[25. Benjaminis Ursini | Sprottavi Silesi | in Electorali Brandenburgico Gymnasio | Vallis Joachimicae, | Cursus Mathematici |

Practici | Volumen Primum | continens | Illustr. Et Generosi DN. | DN. Johannis Neperi | Baronis Merchistonij &c. | Scoti. | Trigonometriam Loga- | rithmicam | Usibus discentium accommoda- | tam cum Gratia Et Privilegio | Typisq: Exscriptam | Coloniae sumtibus Martini Guthij, | Anno CIↃIↃCXVIII.] 1618

8°; A, B, C, Aa-Hh.

The earliest foreign table; published at Berlin. Contains: Dedication and Preface to Dr Abraham, Baron and Burgrave of Dohna, in which reference is made to the method of Prosthaphæresis of Wittich and Buergi; 'Trigonometria Logarithmica Jo. Neperi'; Table of Proportional Parts; Table of Logarithmic Sines, etc., in the form:

Grad. 30.			+ | −			
	Sinus.	Logarithm.	Differ.	Logarit.	Sinus.	
0	50000	69315	54931	14384	86163	60

and a Table of Errata.

In the British Museum, 529. *a*. 9.

26. Beni. Ursini | Mathematici Electora- | lis Brandenburgici | Trigonometria. | cum magno | Logarithmor. | Canone | Cum Privilegio | Coloniae | Sumptib. M. Guttij tipis | G. Rungij descripta. | CIↃ DC XXV.

[Copperplate by Petrus Rollos.]

4°; size $5\frac{1}{2} \times 7\frac{1}{2}$ inches.

Napier's canon is introduced on p. 131; after the treatise on trigonometry comes the logarithmic canon, with a separate title, as follows:—

Benjaminis Ursini | SprottavI SilesI | Mathematici Electoralis Brandenburgici | Magnvs Canon | Triangulorum | Logarithmicus; | Ex Voto & Consilio | Illustr. Neperi, p.m. | novissimo, | et sinu toto 100000000 ad scrupulor. | secundor. decadas | usq́; | Vigili studio & pertinaci industriâ | diductus.

. . . [An extract from Kepler, *Harmonices*, Lib. IV, cap. vii, p. 168] . . .

Coloniae, | Typis Georgij Rungij, impensis & sumptibus Martini Gutij | Bibliopolae, Anno M. DC. XXIV.

Same size and format. The logarithms are Napier's, but represent a fresh calculation, being carried to one place further, and freed from his last place errors; natural sines and tangents—called differentials—with their logarithms, and differences are given for every 10″ of the quadrant. Except for the addition of difference columns, the arrangement is the same as in the last work.

Lent by the UNIVERSITY OF EDINBURGH.

27. Arithmetica | Logarithmica | sive | Logarithmorum Chiliades Triginta, Pro | numeris naturali serie crescentibus ab vnitate ad | 20,000: et a 90,000 ad 100,000. Quorum ope multa | perficiuntur Arithmetica problemata et Geometrica. | Hos Nvmeros Primvs | Invenit Clarissimvs Vir Iohannes Nepervs Baro Merchistonij; eos autem ex eiusdem sententia mutavit, eorumque ortum et vsum illustravit Henricvs Briggivs, | in celeberrima Academia Oxoniensi Geometriae | professor Savilianvs. | Deus Nobis Vsvram Vitae Dedit | Et Ingenii, Tanqvam Pecvniae, | Nvlla Praestitvta Die. |

Londoni, | Excudebat Gulielmus | Iones. 1624.

4°; size $12 \times 7\frac{3}{4}$ inches.

Contains: Dedication to Charles, Prince of Wales, Preface to the Reader, in which the author's decimal notation is explained; also the author's visit to Napier related, and how he found that Napier had already observed the advantage, which had occurred to Briggs, of employing logarithms to base 10; Table of Errata; and Introduction, a-m, pp. 88. There follows in this volume the same in English, printed by George Miller, 1631, and properly belonging to the pirated edition of Vlacq (*q.v.*). The table proper is in 6°, A-R4 containing numbers up to 20000 and Hhhh - Oooo; * containing 90000 to 100000.

The arrangement is the following :—

Chilias Prima

Num. Absolu.	Logarithmi.	Num. Absolu.	Logarithmi.	Num. Absolu.	Logarithmi.
1	0, 00000, 00000, 0000	34	1, 53147, 89170, 4226 1258, 91273, 0802	67	1, 82607, 48027, 0082 643, 41100, 0542
2	0, 30102, 99956, 6398 17609, 12590, 5560	35	1, 54406, 80443, 5028 1223, 44564, 1701	68	1, 83250, 89127, 0624 634, 01780, 3102
3	0, 47712, 12547, 1966 12493, 87366, 0834	36	1, 55630, 25007, 6729 1189, 92232, 9971	69	1, 83884, 90907, 3726 624, 89492, 7700

Briggs's method is the following:—taking the number 10 and extracting the square root 54 times in succession, a series of numbers and corresponding logarithms is obtained to 33 figures, of which the last differ from 1 and from 0 respectively by quantities beginning with 16 ciphers. Now $2^{10}=1024$, and by interpolation among the numbers found it is seen that

$$\log 1{\cdot}024 = {,}\ 01029,\ 99566,\ 39811,\ 95265,\ 27744,$$

whence

$$\log 1024 = 3,\ 01029,\ 99566,\ 39811,\ 95265,\ 27744,$$

and

$$\log 2 = {\cdot}30102,\ 99956,\ 63981,\ 19521,\ 52774.$$

Similarly log 6 is found from the equation $6^9=10077696$, and log 3 from log 6 − log 2, and so other primes. The steps get somewhat less laborious as they proceed. The introduction proceeds with various applications of logarithms to arithmetical and geometrical calculation.

Lent by the Royal Observatory.

28. Another copy of the same, with inscription on the front fly-leaf :

Hunc mihi donavit Henricus Briggius anno 1625,

and on the title-page :

Rob: Naper

and

Rob: Simson. M. DCC. XXXIII.

Lent by the UNIVERSITY OF GLASGOW.

29. Arithmetica | Logarithmica, | Sive | Logarithmorum | Chiliades Centum, Pro | Numeris naturali serie crescentibus | ab Vnitate ad 100000. | Vna Cvm | Canone Triangvlorvm | Sinuum, Tangentium & Secantium, | Ad Radium 10,00000,00000, & ad singula | Scrupula Prima Quadrantis. | Quibus Novvm Traditvr Compendivm, Quo Nvl- | lum nec admirabilius, nec utilius solvendi pleraque Proble- | mata Arithmetica & Geometrica. | Hos Nvmeros Primvs Invenit | Clarissimus Vir Iohannes Nepervs Baro | Merchistonij : eos autem ex ejusdem sententiâ mutavit, eorum- | que ortum & usum illustravit Henricvs Briggivs, | in celeberrimâ Academiâ Oxoniensi Geome- | triae Professor Savilianus. | Editio Secunda aucta per Adrianvm Vlacq Goudanum. | Deus Nobis Vsvram Vitae Dedit Et Ingenii, | Tanqvam Pecvniae, Nvlla | Praestitvta Die. | Goudae, | Excudebat Petrus Rammasenius. | M. DC. XXVIII. | Cum Privilegio Illustr. Ord. Generalium.

6º ; size $13 \times 8\frac{1}{2}$ inches.

Contains: Title, Preface to the Reader by Vlacq, Errata, 4 pp. 'De Usu Canonis,' Briggs's Introduction (excepting chapters xii, xiii, which the reduction to 10 digits permits to be condensed into two short notes), $*_{*}*$, a-g4 (pp. 1-79); 'Chiliades Centum Logarithmorum Pro Numeris Ab Unitate ad 100000.' A-Z, Aa-Zz, Aaa-Kkk. The arrangement is :

Chilias 20.			Chilias 21.			Chilias 21.		
Num.	Logarithmi.	Differ.	Num.	Logarithmi.	Differ.	Num.	Logarithmi.	Differ.
19951 19952	4, 29996, 46686 4, 29998, 64361	2, 17675	20001 20002	4, 30105, 17098 4, 30107, 34229	2, 17131	20051 20052	4, 30213, 60370 4, 30215, 76959	2, 16579

and 'Canon Triangulorum, sive Tabula Artificialium Sinuum, Tangentium & Secantium, Ad Radium 10,00000,00000 & ad singula Scrupula Prima Quadrantis,' Lll-Sss. The arrangement is :

30	Sinvs.	Sin. Compl.	Tang.	Tang. Compl.	Secan.	Sec. Compl.	
0	9,69897,00043 21,87385	9,93753,06317 7,29619	9,76143,93726 29,17004	10,23856,06274 29,17004	10,06246,93683 7,29619	10,30102,99957 21,87385	60
1	9,69918,87428	9,93745,76698	9,76173,10730	10,23826,89270	10,06254,23302	10,30081,12572	59

Vlacq designated his great work a second edition of Briggs's *Arithmetica Logarithmica*, though, as Glaisher remarks, he would have been quite entitled to call it a new work. Vlacq cut down Briggs's calculations from 14 figures to 10, calculated the additional chiliads from 20000 to 90000, and the logarithms of sines, tangents, and secants. This work and the *Trigonometria Artificialis* (below) became the fountain for all except two or three subsequent tables. The sines, etc., of which he gives the logarithms were derived from the tables of Rheticus.

This copy exhibited belonged to Charles Babbage.

Lent by the Royal Observatory.

30. Arithmetique | Logarithmetiqve | . . . | Ces Nombres Premierement | sont inventez par Iean Neper Baron de | Marchiston : Mais Henry Brigs Professeur de la | Geometrie en l'Vniversité d'Oxford, les a | changé & leur Nature, Origine, & | Vsage illustré selon l'inten- | tion du dit Neper. | La description Est Traduite Dv Latin En | François, la premiere Table augmentée, & la seconde | composée par Adriaen Vlacq. | . . . A Goude,

chez Pierre Rammasein. | M. DC. XXVIII. | Avec Privilege des Estats Generaux. |

This is the same work as the last, with the titles and introduction translated into French.

Lent by Professor R. A. SAMPSON.

31. Henrici Briggii | Tafel | van Logarithmi, | voor De Ghetallen Van | een, tot 10000. |
Ter Goude. | By Pieter Rammaseyn, Boeck-vercooper, indc Cor- | te Groenendal, | int Duyts vergult ABC. | MDCXXVI.

8º; size 7×4½ inches.

There are two works bound together, the second without title. The first (A-M2) contains Briggs's logarithms for numbers from 1 to 10,000 to ten decimals, with difference columns; the second (A-F6) contains Vlacq's logarithmic sines, cosines, tangents, cotangents, secants, and cosecants to seven decimals, the arrangement being:

30. GRAD.

M.	SINVS.	Sinus Compl.	TANG.	Tangens Compl.		Ar. Com. Sinus.	Ar. Com. Sin. Com.
0	9, 6989700	9, 9375306	9, 7614394	10, 2385606	60	, 3010300	, 0624694
	⋮						

(31 lines to page.)

59 Grad.

Lent by the ROYAL OBSERVATORY.

32. Nievwe | Telkonst, | inhovdende de | Logarithmi voor de Ghe- | tallen beginnende van 1 tot 10000, ghemaeckt | van Henrico Briggio Professor | van de Geometrie tot Ocxfort. | Mitsgaders | De Tafel van Hoeckmaten ende Raecklijnen door | het ghebruyck van Logarithmi, de Wortelzijnde van 10000,0000 deelen, gemaeckt van Edmund. Gun- | tero, Professor van de Astronomie tot Londen. |
Welcke ghetallen eerst ghevonden zijn van | Ioanne Nepero Heer van Merchistoun: Ende'tgebruyc daer van is met eenige Arithmeti- | sche, Geometrische ende Sphaerische Exempelen |

cortelijk aenghewesen, | Door Ezechiel de Decker, Rekenmeester, ende | Lautmeter residerende ter Goude | Ter Govde. | By Pieter Rammaseyn, Boeck-Verkooper inde | corte Groenendal, int vergult ABC. 1626 | Met Previlegie voor thien Iaren.

8°; size $7 \times 4\frac{1}{2}$ inches.

Similar to last, so far as the first part is concerned. The second part has two title-pages, the first reading | 'Edmvndi Gvnteri | Tafel | Van Hoeck-maten ende | Raeck-lijnen, den Wortelzijnde van | 10000,0000 deelen' |, and the second, 'Vne Table | Logarithmetiqve : | en | Laqvelle sont conpris | les Logarithmes des Sinus & Tangentes | de tous les Degrez & Minutes du | quart de Cercle; | Le Logarithme dv | demy-dia- | metre du Cercle estant posé | 10000,0000.' |

Only sines and tangents are given, the arrangement being:

M.	Sin. 29.		Tan. 29.		
1	9685,7990	9941,7492	9744,0498	10255,9501	59
	⋮				

(30 lines.)

The figure type is different in the two parts.

Lent by University College, London.

33. Logarithmicall | Arithmetike. | Or Tables of | Logarithmes For | Absolute Numbers From An | unite to 100000; as also for Sines, Tangentes and Secantes for | every Minute of a Quadrant: with a plaine description of their use in | Arithmetike, Geometrie, Geographie, Astro- | nomie, Navigation, &c. | These Numbers were first invented by the most excellent Iohn | Neper Baron of Marchiston, and the same were transformed, and the foundation and | use of them illustrated with his approbation by Henry Briggs, Sir Henry Savils | Professor of Geometrie in the Vniversitie of Oxford. The uses whereof were written in | Latin by the Author himselfe, and since his death pub-

lished in English by diverse of his friends according to his mind, for the benefit | of such as understand not the Latin tongue. | Deus Nobis Vsvram Vitae Dedit, Et Inge- | nii, Tanquam Pecvniae, | Nvlla Praestitvta Die. | London. | Printed by George Miller. 1631. |

There is a note by Lord Lindsay inserted in this volume : ‘ This work, given to Briggs on the title-page, is in reality a copy of Vlacq of the Gouda Edition of 1628. After the death of Briggs in 1631 one G. Miller bought up a certain number of copies of Vlacq's *Arithmetica Logarithmica*, printed a new title and introduction in *English*, affixed them to the Vlacq, and issued it to the world as a reprint of Briggs's work with the same title.'

The introduction, A-H (pp. 1-62), is in 4°, and on different paper from the table, which is in 6°. It contains simple examples of the practical application of the tables to questions of arithmetic, geometry, and astronomy, with a short catalogue of stars and some other details, for the use of navigators.

This copy belonged to Babbage.

Lent by the ROYAL OBSERVATORY.

34. Trigonometria | Britannica | sive | De Doctrina Triangulorvm | Libri Dvo. | Quorum Prior continet Constructionem Cano- | nis Sinuum Tangentium & Secantium, unà cum Lo- | garithmis Sinuum & Tangentium ad Gradus & Gra- | duum centesimas & ad Minuta & Secunda Centesimis respondentia : | A Clarissimo Doctissimo Intergerrimoque Viro Domino Henrico | Briggio Geometriae in Celeberrima Academia Oxoniensi | Professore Saviliano Dignissimo, paulo ante inopinatam | Ipsius e terris emigrationem compositus. | Posterior verò usum sive Applicationem Ca- | nonis in Resolutione Triangulorum tam Planorum | quam Sphaericorum e Geometricis fundamentis | petitâ, calculo facillimo, eximiisque compendiis exhibet : | Ab Henrico Gellibrand Astronomiae in Collegio | Greshamensi apud Londinenses Professore constructus. | . . .

| Goudae, | Excudebat Petrus Rammasenius. | M. DC. XXXIII. | Cum Privilegio. |

4^o; $13 \times 8\frac{1}{2}$ inches.

Contains: Title, Gellibrand's Dedication to the Electors to the Savilian Chair, his Preface (from which we learn that Vlacq took upon himself the cost of printing the work), 'Trigonometriae Britannicae Liber Primus,' A-H2 (pp. 1-60); on the division of the circle by square and cube roots, including a table of natural sines for intervals $0^\circ{\cdot}625$ throughout the quadrant, to 19 places of decimals, and of log. sines for intervals $1^\circ{\cdot}250$, to 14 places; ditto, 'Libri Secundi Pars Prima De Triangulis Planis,' $H3^1$-$K2^2$ (pp. 61-76); ditto, ditto, 'Pars Secunda De Triangulis Sphaericis,' $K3^1$-$O3^2$ (pp. 77-110); 'Canones Sinvvm Tangentivm Secantivm et Logarithmorum pro Sinvbvs & Tangentibvs, ad Gradus & Graduum Centesimas, & ad Minuta & Secunda Centesimis respondentia,' in 6^o, a-$z4^1$, arranged thus:

30. GRAD.

Centesimae.	Sinus.	Tangentes.	Secantes.	Logarithmi Sinuū.	Log. Tangent.	M:S.
0	50000,00000,00000 15,11423,30817	57735,02692 23,27340	115470,05384 11,63846	9,69897,00043,3602 13,12607,2248	9,76143,93726 17,50320	0:0
1	50015,11423,30817	57758,30032	115481,69230	9,69910,12650,5850	9,76161,44046	0:36
⋮						

(34 lines to page.)

The work retains the division of the quadrant into 90 degrees, but divides each degree centesimally.

Lent by the ROYAL OBSERVATORY.

35. Trigonometria | Artificialis: | sive | Magnvs Canon | Triangvlorum | Logarithmicvs, | Ad Radium 100000,00000, & ad dena Scrupula Se- | cunda ab Adriano Vlacco | Goudano Constructus. | Cui Accedunt | Henrici Briggii Geometriae Professoris in | Academiâ Oxoniensi P. M. Chiliades Logarithmorum | Viginti

pro numeris naturali serie crescentibus ab Vnitate ad | 20000, unde Canon iste deductus est. | Ad quorum usum illustrandum, | additus est Liber Secundus | Trigonometriae Britannicae, in quo Tri- | angula Plana & Sphaerica calculo fa- | cillimo, eximiisq. Compendijs è Geometricis | fundamentis petitis resolvuntur. | . . . | Goudae, | Excudebat Petrus Rammasenius | Anno M. DC. XXXIII. | Cum Privilegio. |

4°; size 13×8½ inches.

+, A-G2 contains Title, Dedication to the Count Palatine, Preface to the Reader, in which Vlacq tells how he consented to print Briggs's *Trigonometria Britannica*, and considered its centesimal division much superior to the sexagesimal, but because many did not prefer it, and were unwilling to change their habits, and because the differences at the beginning of Briggs's table were excessively great, he published the present work which he had finished three years before; there follows 'Trigonometria Artificialis,' which is a reprint with a few changes of the 'Liber Secundus' on solution of plane and spherical triangles of Briggs's introduction to the *Trigonometria Britannica*, 'Magnus Canon Triangulorum, continens Logarithmos Sinuum et Tangentium ad dena Scrupula Secunda,' in 6°, a-z, arranged as under:

30 M.	30 S.	SINVVM. Logarithmi.	Differ.	Sin. Compl. Logarithmi.	Differ.	TANGEN. Logarithmi.	Differ. Communes.	Tang. Compl. Logarithmi.		
0	0	9,69897,00043		9,93753,06317		9,71643,93726		10,23856,06274	0	60
			3,64667		1,21569		4,86236			
	10	9,69900,64710		9,93751,84748		9,71648,79962		10,23851,20038	50	
					(61 lines to page.)					
10	0								0	50
		Sin. Compl.		SINVVM.		Tang. Compl.		TANGEN.	S.	M.

There follows: 'H. Briggii Tabvla sive Chiliades Viginti Logarithmorum pro numeris naturali serie crescentibus ab Vnitate ad 20000.' A-M2^2. Briggs's table is cut down to ten places, and the differences are in separate columns in place of being interlinear. A table of errata follows. Some of these are misprints carried forward from Briggs.

Lent by the Royal Observatory.

36. Joannis Kepleri | Imp. Caes. Ferdinandi II. | Mathematici | Chilias | Logarithmorum | Ad Totidem Numeros Rotundos, | Praemissâ | Demonstratione Legitima | Ortus Logarithmorum eorumq; usus | Quibus | Nova Traditur Arithmetica, Seu | Compendium, Quo Post Numerorum Notitiam | nullum nec admirabilius, nec utilius solvendi pleraq; Problemata | Calculatoria, praesertim in Doctrina Triangulorum, citra | Multiplicationis, Divisionis, Radicumq; extractio- | ni in Numeris prolixis, labores mole- | stissimos. | AD | Illustriss. Principem & Dominum | Dn. Philippum | Landgravium Hassiae, Etc. | Cum Privilegio Authoris Caesareo. |
Marpurgi, | Excusa Typis Casparis Chemlini. | CIↃ IↃC XXIV. |

4°; A-O; size $7\frac{1}{2} \times 6\frac{1}{2}$ inches.

37. Joannis Kepleri, | Imp. Caes. Ferdinandi II. | Mathematici, | Supplementum | Chiliadis | Logarithmorum, | continens | Praecepta De Eorum Usu, | Ad | Illustriss. Principem | Et Dominum, | DN. Philippum Land- | Gravium Hassiae, &c. |
Marpurgi, | Ex officinâ Typographica Casparis Chemlini. | CIↃ IↃ CXXV. |

4°; P-Z, Aa-Dd; same size.

We learn from the explanatory introduction to the second part that the work was undertaken to supply a formal demonstration of logarithms, presumably in ignorance of Napier's 'Construction'; and also to emphasise the fact that logarithms are purely of arithmetical origin, and that Napier's presentation of them in connection with sines is not a necessary one. The arrangement is the following :

Arcus. Circuli cum differentiis.	Sinus. Seu Numeri absoluti.	Partes vicesima quarta.	Logarithmi. Cum differentiis.	Partes Sexagenaria.
4·19			165·43	
37·13·46	60500·00	14·31·12	50252·68 +	36·18
4·19			165·15	
37·18·5	60600·00	14·32·38	50087·53	36·22

The second column contains what we should now call the argument of the table, and contains 1000 entries ; the third and fifth columns contain the same, but divided so that the full quantity 100000·00 of the second column corresponds to $24^{h.}\ 0^{m.}\ 0^{s.}$ of the third, and to 60′ of the fifth ; the first column contains the arc, if the second be treated as a sine ; the fourth contains the logarithms ; these are Napier's logarithms, but are apparently the result of an independent calculation, and are more correct in the last place. One rule arrived at by Kepler may be mentioned (*Chilias*, p. 43) : it may be put in the form that the difference of the logarithms of two consecutive integers is equal to the arithmetic mean of their reciprocals ; this is equivalent to the equation

$$\log_e \frac{N+1}{N} = \frac{1}{N} - \frac{1}{2}\frac{1}{N^2} \cdot \cdot \cdot$$

Lent by the Royal Observatory.

VI.—THE DISCOVERY OF LOGARITHMS BY JOBST BUERGI

The following passage occurs in Kepler's *Tabulae Rudolphinae*, 1627—Caput iii, p. 11—speaking of his own continued use of the logarithms employed in his *Chilias*—that is, of Napier's logarithms, and the relative advantage of Briggs's, he says that a system, such as we now call antilogarithms, would be still more advantageous, because it would give whole numbers throughout and not merely for powers of 10, and that algebraic indices (*apices logistices*) would point to this, and continues: 'qui etiam apices logistici Justo Byrgio multis annis ante editionem Neperianam, viam praeiverunt, ad hos ipsissimos Logarithmos. Etsi homo cunctator et secretorum suorum custos, foetum in partu destituit, non ad usus publicos educavit.'

Buergi's manuscripts are at the Observatory at Pulkowa, and, according to a note by Lord Crawford (on Struve's authority) in the Royal Observatory copy of Kepler's *Tabulae Rudolphinae*, none are of date later than 1610; but his work described below was not published until 1620, when that of Napier was already famous. Buergi has suffered in consequence from undeserved neglect. His improvements in the method of Prosthaphæresis show otherwise how his powerful and ingenious mind was engaged upon improvements in the means of calculation.

38. Tabulae | Rudolphinae, | Qvibvs Astronomicae Scientiae, Tempo- | rum longinquitate collapsae Restauratio continetur; | A Phoenice illo Astronomorum | Tychone | . . . | Primum Animo Concepta. | . . . | Tabulas ipsas . . . perfecit | Ioannes Keplerus, | . . .
. . . Ulmae . . . Anno M DC XXVII.

4°; 9×13 inches.

The references to logarithms and the work of Buergi are on p. 11.

39. Aritmetische und Geometrische Progress | Tabulen, sambt gründlichem unterricht, wie solche nützlich | in allerley Rechnungen zugebrauchen, und verstanden werden sol. |
Gedruckt, In der Alten Stadt Prag, bey Paul | Sessen, der Löblichen Universitet Buchdruckern, Im Jahr 1620. |

The title-page and first page of numbers are reproduced in facsimile in Plates XII and XIII. In the centre of the title-page a device of two circles, the outer one containing numbers in red, the inner the corresponding antilogarithms in black, and within these | 'J B | Die ganze Rote Zahl | 230270022. | Die ganze Schwarze Zahl | 1000000000.' |

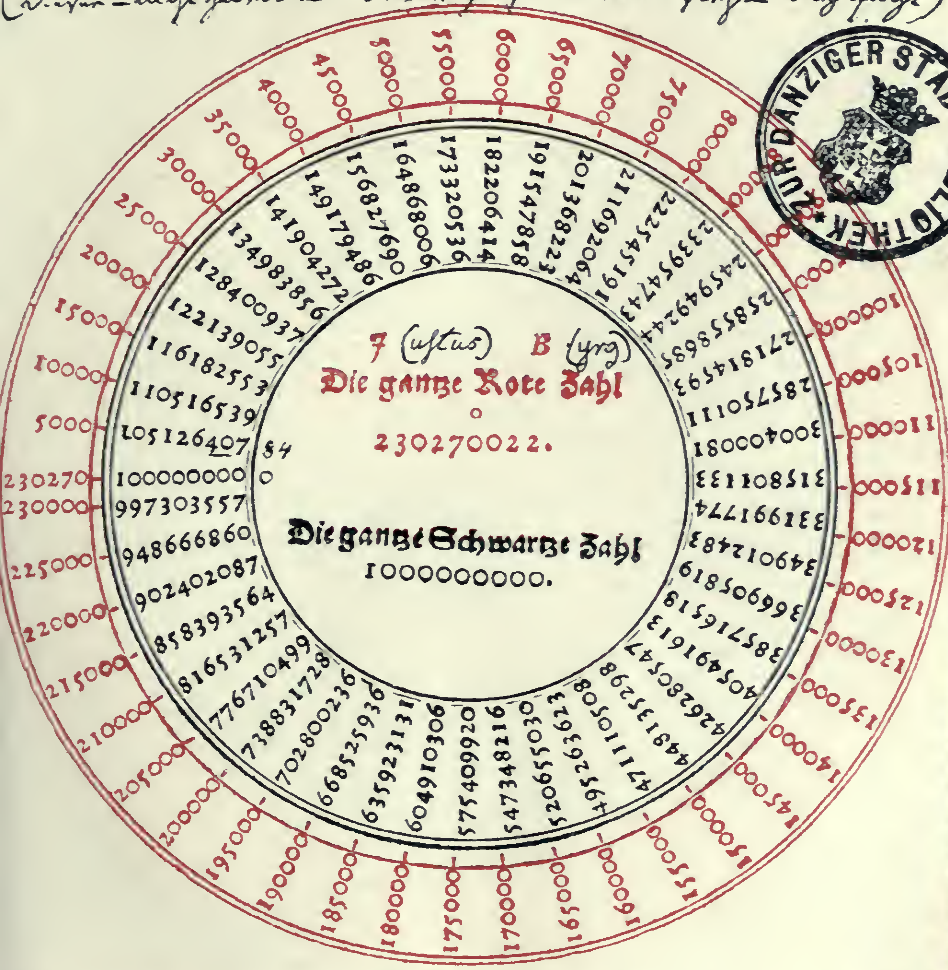
4°; size $6 \times 7\frac{1}{4}$ inches; A-H2, followed by ten pages in MS. containing the Preface and Explanations. We owe it to the generosity of the Town Library of Dantzig that we were able to exhibit this exceedingly rare work of Buergi's, rendered unique by the addition of the MS. of his introductory matter, which was never printed. The work itself is singularly clear and simple; it is founded strictly upon the ideas of indices; taking the number 1·00010000—the decimal point supplied as we should now use it—and associating it with the index 1·0, he raises it successively to the powers of index 2·0, 3·0, 4·0, and so on; the number increases and increases by small steps until with index between 23027·0022 and 23027·0023 it reaches 10·000,000,00. The table is therefore strictly a table of antilogarithms with base 1·0001, the logarithms being tabulated unit by unit to five digits, and the corresponding numbers being given to eight places of decimals. The logarithm is printed in red along the top and down the edge of the page, and called by Buergi *Die Rothe Zahl* simply. The arrangement, which contains a superfluous cypher to all the logarithms, is shown in the reproduction of the first page.

The device upon the title-page is also reproduced and is a summary of the contents of the book, and is, in fact, very like a circular slide rule. As regards the accuracy of the work, the logarithm (to base 10) of 1·0001 being ·0000 4342 7276 86, if we multiply this by 23027·0022 we find ·9999 9999 97, which agrees with Buergi's statement at the end of the tables that 23027·0022 is slightly in defect; thus his calculations have been carried out with sufficient

Aritmetische vnd Geometrische Progress

Tabulen/sambt gründlichem vnterricht/wie solche nützlich in allerley Rechnungen zugebrauchen/vnd verstanden werden sol.

(Dieser – nicht gedruckte – Unterricht ist ein Manuscript beigefügt)



Gedruckt/ In der Alten Stadt Prag/ bey Paul Sessen/der Löblichen Universitet Buchdruckern/ Im Jahr / 1 6 20.

PLATE XIII

	0	500	1000	1500	2000	2500	3000	3500
0	100000000	100501227	101004966	101511230	102020032	102531384	103045299	1035617[illegible]
10	10000	11277	15067	21381	30234	41637	55603	....7214[illegible]
20	20001	[illegible]	[illegible]	[illegible]	40437	51891	65909	[illegible]
30	30003	31380	35271	41687	50641	62146	76216	92861
40	40006	41433	45374	51841	60846	72402	86523	103603221
50	50010	51487	[illegible]	[illegible]	71052	....8266[illegible]	....9683[illegible]	13581
60	60015	61543	65584	72158	81259	92918	10310714[illegible]	23942
70	70021	71599	75691	82309	91467	102603177	....1745[illegible]	34305
80	80028	81656	85799	92468	102101676	13438	27764	....4466[illegible]
90	90036	91714	95907	101602627	11887	23699	38077	....5503[illegible]
100	100100045	100601773	101106017	12787	22098	33961	48391	....6539[illegible]
110	10055	11834	....161[illegible]	22049	32310	44225	58705	....7576[illegible]
120	20066	21895	26239	33111	42523	54489	69021	86132
130	30078	31957	36352	43274	52738	64755	79338	96501
140	40091	[illegible]	46465	[illegible]	62953	75021	89656	103706871
150	50105	52084	56580	63604	73169	85289	99975	17241
160	60120	62150	66696	73770	83386	95557	103210295	27613
170	70136	72216	76812	83938	....9360[illegible]	102705827	[illegible]	37986
180	80153	82283	86930	94106	102203824	16097	....3093[illegible]	....4836[illegible]
190	90171	92351	97045	101704275	14045	26365	41261	58734
200	100200190	[illegible]	[illegible]	14446	24266	....3664[illegible]	....5158[illegible]	....6911[illegible]
210	10210	12491	17289	24617	34488	46915	61910	....7[illegible]
220	20231	22562	27411	34790	....4471[illegible]	57190	72237	....8[illegible]
230	30253	32634	....375[illegible]	44963	54936	67466	82564	103800244
240	40276	42707	47657	55138	65162	77742	92892	10624
250	50300	52782	57782	65313	75388	88020	103303221	....21[illegible]
260	60325	62857	....679[illegible]	75490	85616	98299	....1355[illegible]	31387
270	70351	72933	78085	....8566[illegible]	95845	102808579	23883	41770
280	80378	83011	88162	95846	102306074	18860	34216	....5215[illegible]
290	90406	93189	98291	101806025	16306	....2914[illegible]	....44[illegible]	....6[illegible]
300	100300435	100803168	101308421	16206	26536	39425	54883	....7292[illegible]
310	10465	13248	18552	26387	36765	49708	65219	....8331[illegible]
320	20496	23330	28684	36570	47003	59993	....755[illegible]	[illegible]
330	30528	33412	38817	46754	57237	70279	85893	103904091
340	40562	43496	48950	56939	67473	80566	96232	14481
350	50596	53580	59085	67124	77710	[illegible]	103406571	....2487[illegible]
360	60631	63665	69221	77311	87947	102901144	....1691[illegible]	35265
370	70667	73752	79358	87499	98186	11434	27254	45659
380	80704	83839	89496	97687	102408426	21725	37596	....560[illegible]
390	90742	93927	99635	101907877	18667	32017	....4794[illegible]	....6644[illegible]
400	100400781	100904017	101409775	....1806[illegible]	28905	42310	58285	76846
410	10821	14107	19916	....2826[illegible]	39152	52604	....6[illegible]	....8724[illegible]
420	20862	24199	30058	....3845[illegible]	49396	....6290[illegible]	....7897[illegible]	....9[illegible]
430	30904	34291	40201	....4864[illegible]	59641	73196	89326	10400804[illegible]
440	40948	44384	50345	58841	69887	83493	99674	....1[illegible]
450	50991	54479	60489	69037	80133	....9379[illegible]	103510024	....2884[illegible]
460	61037	64574	70636	79234	90381	103004091	20375	....3924[illegible]
470	[illegible]	74671	80783	89432	102500630	....14[illegible]	[illegible]	....406[illegible]
480	....[illegible]	....8476[illegible]	90931	99631	....1088[illegible]	....2469[illegible]	....4108[illegible]	....600[illegible]
490	91178	94867	101501080	102009831	....2113[illegible]	34995	51435	....7046[illegible]
500	100501227	101004966	11230	20032	31384	45299	61790	80816

precautions, and are correct to the last figure. After the tables comes a manuscript containing Buergi's preface and explanations. This MS. is apparently not in Buergi's hand, but it is supposed that it may be in that of Bramer, his brother-in-law, and a member of his household till 1611. It is fairly well written, but is more easy to follow in the transcription by Dr Gieswald, which is annexed (No. 40, below). In the *Vorrede an den Treuherzigen Leser*, which we should judge to have been written in complete ignorance of Napier's work, Buergi speaks of the labour of multiplication, division, the extraction of roots, etc., and the limited utility of special tables for effecting them, and continues: 'Derowegen ich zu aller Zeit gesucht und gearbeitet habe, general Tabulen zu erfinden, mit welchen mann die vorgenannten Sachen alle verrichten möchte. Betrachtent derowegen die eigenschafft und Correspondenz der 2 progressen alsz der Aritmetischen mit der Geometrischen, das was in der ist Multipliciren, ist in iener nur Addiern und was in der is Diuideren in iener subtrahiern und was in der ist radicem quadratam extrahirn in iener nur ist halbiren, radicem cubicam extrahirn nur in 3 diuidirn . . . und also fort in andern quantiteten, so habe ich nichts nutzlicheres erachtet, alsz diese Tabulen also zu continuiern dasz alle Zahlen so vorfallen in derselben mögen gefunden uerden. . . .' Gieswald quotes very appositely a parallel passage from Stifel's *Arithmetica Integra* (1544), of which the above may very well be a recollection (Lib. i, p. 35): 'Additio in Arithmeticis progressionibus respondet multiplicationi in Geometricis; Subtractio in Arithmeticis respondet in Geometricis divisioni. Divisio in Arithmeticis progressionibus, respondet extractionibus radicum in progressionibus Geometricis. Ut dimidiatio in Arithmeticis respondet extractioni quadratae in Geometricis, . . . et sic de aliis in infinitum.' A perusal of Napier's and Buergi's work suggests that Buergi thought in algebra and Napier in geometry.

With respect to the publication of his work Buergi says: ‘Und ob wol ich mit dieszen Tabulen vor ettlichen Jahren bin umbgang so hat doch mein Beruff von der Edition derselben enthalten. . . .’ This is what Kepler said in other words. There is no date to it, but Bramer, writing in 1630, says the table was calculated ‘vor zwantzig und mehr Jahren.’

Buergi continues with his explanation; he takes the two associated progressions of ‘red’ and ‘black’ numbers:

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	4	8	16	32	64	128	256	512	1024	2048	4096

and shows how various questions, such as the values of 8×64, 32×256, $16384 \div 512$, $128 \times 32 \div 8$, and so on, can be answered by their aid, by simple additions and subtractions. He then goes on to show how to employ his own Progression Tables for the same ends, and how to interpolate in them for numbers that are not given.

Lent by the Town Library of Dantzig.

40. Dr Gieswald; Justus Byrg als Mathematiker und dessen Einleitung in seine Logarithmen. (Bericht über die St. Johannis-Schule, . . . Danzig, 1856. Schnellpressendruck der Wedel'shen Hofbuchdruckerei.) 36 pp.

This contains a great deal of information respecting the writings of Byrg and Bramer and their contemporaries, concluding with a complete transcript of the manuscript of the Preface and Instructions which are bound up with the Dantzig copy of Byrg's tables, printed in red and black figures like the original.

Lent by the Town Library of Dantzig.

VII.—THE GREAT TABLES PRECEDING THE DISCOVERY OF LOGARITHMS

There is abundant evidence of the urgency for improved methods of calculation in the reception of Napier's Rabdology, in the invention of Prosthaphæresis, in Buergi's independent discovery of logarithms, and not least in the great calculating tables of Rheticus, with the additions of Otho and Pitiscus and others. As we gather from Wright's English edition of the *Description*, this sprang from the needs of mariners in navigating ships over long voyages. The collection of calculating books here shown is intended to illustrate the position of the art of calculation at the period of the discovery of logarithms. It will be seen that so late as 1630 an edition of the Trigonometry of Pitiscus was published—intended apparently for the use of sailors—which employs natural sines and makes no use of the method of logarithms.

41. Opus | Palatinum | De | Triangvlis | A | Georgio Ioachimo | Rhetico Coeptvm : | L. Valentinvs Otho | Principis Palatini | Friderici | IV. Electoris | Mathematicvs consvmmavit. | An. Sal. Hvm. | CIƆ. IƆ. XCVI. |

    Plin. Lib. xxxvi, cap. ix | Rerum Naturae Interpreta- | tionem Ægyptiorvm Opera Phi- | losophiae continent Cum Privilegio | Caes. Maies. [Copperplate.]

4°; size 8×14 inches; 2 vols.

The first contains the treatise of Rheticus, *De Triangulis Globi cum Angulo Recto* (pp. 3-140), that of Otho, *De Triangulis Globi sine Angulo Recto Libri Quinque* (pp. 1-341), and Otho's table, *Meteoroscopium . . . , monstrans proportionem singulorum Parallelorum ad Æquatorem vel Meridianum,* which tabulates the quantities cos b sin l, cos b cos l, and arc sin (cos b sin l)—denoting by b, l the latitude and longitude—for every degree of b and of l, together with columns of differences. Volume 2 contains the work of Rheticus, *Magnus Canon Doctrinae Triangulorum Ad Decades Secundorum Scrupulorum Et Ad Partes* 1·000 000 000 0. The first portion gives the trigonometrical functions for every 10″ to ten places of decimals in the following form: taking a right-angled triangle with the given radius as hypotenuse, the perpendicular and base are given, that is the sine and cosine; again taking it as the longer of the two containing sides, the hypotenuse and perpendicular are given, that is the secant and tangent; and finally taking it as the shorter of the two sides, hypotenuse and base are given, that is the cosecant and cotangent. Errata of the table follow. After this comes *De Fabrica Canonis* (pp. 1-103), and finally a different calculation of the third series above, that is cosecants and cotangents to eight places of decimals. This is less accurate than the first table; differences begin to show themselves about 33°, and increase rapidly as the angle diminishes, so that at the beginning only the three first figures are correct. This copy belonged successively to Lalande, Delambre, and Babbage before it came into the hands of Lord Crawford. There are notes on the title-page in the hand of Delambre.

Lent by the Royal Observatory.

42. Georgii Io- | achimi Rhae- | tici | Magnus Canon | Doctrinae | Triangulorvm Ad | Decades Secvn- | dorvm Scrupvlorvm | Et Ad Partes 10000000000. | Recens emendatus à Bartholomæo Pitisco Silesio. | Addita est brevis commonefactio de fabrica & | vsu huius Canonis. Quae est summa do- | ctrinae, & quasi nucleus, totius | operis Palatini. |
Canon hic, | unà cum brevi commonefactione de eius fabrica & vsu, | etiam separatim ab opere Palatino venditur. |
In Bibliopoleis Harnischiano.

This is a reissue of the Great Canon of the six trigonometrical functions, to ten places of decimals and to 10″, without change. The table of errata at the end is given without addition or correction of its most obvious misprints. After the table come the very rare sheets:

Bartholomaei Pitisci | Grünbergensis Silesii | Brevis Et Per- | spicua commo- | nefactio | De Fabrica Et | Vsv Magni Cano- | nis doctrinae Triangulorum Geor- | gii Ioachimi Rhetici. | Neostadii, | Typis Nicolai Schrammii | MDCVII.

The copy belonged to Babbage.

Lent by the ROYAL OBSERVATORY.

43. Thesaurus Mathematicus | sive | Canon Sinuum | Ad Radium | 1.00000.00000.00000. | Et Ad Dena Qvaeqve Scrvpvla | secunda Quadrantis: | Una Cum Sinibus Primi | Et Postremi Gradus, Ad | Evndem Radivm, Et Ad Singvla | scrupula secunda Quadrantis: | Adivnctis Vbiqve Differentiis Primis Et Se- | cundis; atq; vbi res tulit, etiam tertijs. | Iam Olim qvidem Incredibili Labore | Et sumptu à Georgio Joachimo Rhetico supputatus: | At Nvnc Primvm In Lvcem Editvs, | & cum viris doctis communicatus | A | Bartholomæo Pitisco | Grunbergensi Silesio. | Cvivs Etiam Accesservnt: | I. Principia Sinuum, ad radium, 1.00000.00000.00000.00000. | 00000. quàm accuratisimè supputata. | II. Sinus decimorum, tricesimorum & quinquagesimorumq; scrupulorum secundorum per prima & postrema 35. scrupula prima, ad radium, 1.00000.00000. 00000.00000.00. |
Francofurti | Excudebat Nicolaus Hoffmannus, sumptibus | Jonae Rosae Anno | CIƆ. IƆ.C. XIII.

6°; size $9\frac{1}{2}\times 14$ inches.

After an interesting preface to the reader by Pitiscus on the use of tables computed to this degree of accuracy, the table begins with sines and cosines for every 10″, to 15 places of decimals, with first, second, and third differences; where these change sign they are written, *e.g.* 3 or 0−3. After this comes:

Sinus Primi | Et Postremi | Gradus | Ad Singula Scru- | pula

Secunda, | Et Ad Partes Radij | 1.00000.00000.00000. | Unà cum differentijs primis & secundis. | Primvs et Postremvs Numervs Vnivs | cuiusq ; paginae gradus denotat : Secundus, scrupula prima : | Reliqui, scrupula secunda. |
Francofurti | Typis Nicolai Hoffmanni, Impensis | Ionae Rosae, Anno | CIƆ. IƆ. XIII.

That is to say, it is a table of sines and cosines to 15 places of decimals, for every second of the first degree. There follows :

Principia Sinuum | Ad Radium | 1.00000.00000.00000.00000. 00000. | Per Analysin Algebraicam | Inventa : Et Per Synthesin Contrariam Demon- | strata : perq́, digitos multiplicata, Et probatione novenaria communita ; | atq ; adea in tabulas ad compendia calculi vtilis- | simas redacta : | Auctore | Bartholomæo Pitisco | Grunbergensi Silesio. | Accessere tabulae consimiles, ex Sinibus arcuum. x. & xx. scrupulorum secundorum, & complementorum eorun- | dem factae. | Item duo exempla compendiosi calculi : unum multiplica- | tionis, alterum divisionis : ex tabulis illis. |
Francofurti | Typis Nicolai Hoffmanni, Impensis Ionae Rosae. | Anno CIƆ. IƆ C. XIII.

The *Principia* are verifications of values given to 25 places of decimals of, in effect, sin 30°, sin 15°, sin 5°, sin 1°, sin 30′, sin 10′, sin 5′, sin 1′, sin 30″, sin 10″, sin 5″, sin 1″, by means of formulæ $\sin 3a = 3 \sin a - 4 \sin^3 a$, and the like, drawn from his *Trigonometry* ; then follow the first ten multiples of these numbers, with a numerical check ; and two examples of multiplication and division with 25 digits—these are worth notice ; contracted methods are used ; it will be remarked that logarithms would not assist calculations of this number of digits. After these come :

Sinus Decimorum, | Tricesimorum, Et Quin- | quagesimorum Quorumque | scrupulorum secundorum, in prioribus triginta quinq ; scrupulis primis contentorum. | Una cum Sinibus Com- | plementorum | Ad Radium. | 1.00000.00000.00000.00000.00 |

Additis Differentiis Primis, | secundis, tertiis, quartis, quintis. | Ex Svppvtatione | Bartholomæi Pitisci | Grunbergensis Silesii.| Francofurti | Excudebat Nicolaus Hoffmannus, sumptibus | Jonae Rosae Anno 1613.

That is, a table of sines for every 20″ of the first 35′ (but beginning at 10″), to 22 places of decimals, with differences.

A number of interesting MSS. are bound with this copy, before and after the tables, among them an extract from the will of Lalande, bequeathing it to Delambre. The MSS. at the end, in the hand of Delambre, are for the most part studies of the works of Rheticus, Otho, and Pitiscus.

Lent by the Royal Observatory.

44. Canon | Triangvlorvm | Emendatissimvs, | Et Ad Vsvm Accom- | modissi- | mus; | Pertinens Ad Trigono- | metriam | Bartholomæi Pitisci | Grvnbergensis | Silesii. | s.l. | MD. C. VIII.

4°; size $6 \times 7\frac{1}{2}$ inches.

A table of sines, tangents, and secants, and the same for the complementary angle; the tabulation is for every 1″ up to 1′, for every 2″ up to 10′, for every 10″ up to 1°, and thereafter for every 1′, with proportional parts for 10″. Generally the entries run to 7 places of decimals, but for the early secants and cosines 10 are given, while for the cosecants and cotangents 8 significant figures are given, or say 6 places of decimals. At the end are an advertisement of the *Trigonometry* of Pitiscus, and errata.

Lent by the Royal Observatory.

Another edition of the same with Hoffmann's and Jonas Rosa's imprint, and date 1612. It contains a table of errata at the end.

Lent by the Royal Observatory.

45. Bartholomæi | Pitisci Grunbergensis | Silesij | Trigonometriae | Sive | De dimensione Triangulorum | Libri Quinque, . . . | Editio Tertia | . . .
Francofurti. | Typis Nicolai Hofmanni: | Sumptibus Ionae Rosae. | M. DC. XII. [Copperplate.]

4°; size $6\frac{1}{2}\times8\frac{1}{4}$ inches.

This work, with applications to astronomy, geodesy, navigation, architecture, etc., gives an illustration of the necessities of calculation and the knowledge of practical experts at this period.

Lent by the Royal Observatory.

46. Trigonometry: | Or The | Doctrine | Of | Triangles. | First written in Latine, by | Bartholomew Pitiscvs | of Grunberg in Silesia, and now | Translated into English, | by Ra: Handson. | Whereunto is added (for the Mariners | use) certaine Nauticall Questions, to- | gether with the finding of the Variation of | the Compasse. All performed Arith- | metically, without Map, Sphære, | Globe or Astrolabe, | by the said R. H. |
Printed by T. F. for G. Hurlock | near *Magnus* Corner. | s.l., s.a.

8°; size $5\frac{1}{2}\times7\frac{1}{4}$ inches.

This book is shown as illustrating that at the date of its publication logarithms—to which reference is made in the *Epistle Dedicatorie* to the Masters, etc., of Trinity House—had not yet wholly displaced the older methods. The date is fixed by the second work bound up in the same volume:

A | Canon Of | Triangles: | Or, | The Tables, | Of | Sines, Tangents, and Secants, | The Radivs asumed | to be 100000. |
London. | Printed for Iohn Tap. 1630.

It is a five-figure table of the six natural functions to single minutes of arc.

Lent by the Royal Observatory.

VIII.—THE METHOD OF PROSTHAPHÆRESIS

The word Prosthaphæresis means Addition and Subtraction (*προσθέσις, ἀφαίρεσις*) and was used in this general sense, *e.g.* by Tycho, as denoting an astronomical equation, like the equation of the centre, which—as we should now put it—changed its sign. It is used with like generality by Napier (*Constructio*, p. 20) to refer to his own method: 'Ex his praelibatis judicent eruditi quantum emolumenti adferent illis logarithmi: quandoquidem per eorum additionem multiplicatio, per substractionem divisio, per bipartitionem extractio quadratae, per tripartitionem cubicae, et per alias faciles prosthaphaereses omnia graviora calculi opera evitantur . . .' It is, however, particularly used to denote a method of effecting the multiplication of two sines, by means of the theorem

$$\sin a \sin b = \tfrac{1}{2}\{\sin(90^\circ - a + b) - \sin(90^\circ - a - b)\}.$$

Delambre (*Astronomie du Moyen Age*, p. 112) shows that such an equation was used by Ibn Yunis (A.D. 1000) to facilitate calculation, but upon the authority of Longomontanus, *Astronomica Danica*, 1622, p. 10, it was invented as a systematic adjunct to astronomy in 1582 by Tycho and Wittich of Breslau, who was at that time Tycho's assistant. From Wittich it reached Buergi, who generalised it, and by its aid reduced the equation

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

to the remarkably convenient form

$$\cos c = \tfrac{1}{2}[\cos(a-b) + \cos(a+b) + \cos(C-x) + \cos(C+x)]$$

where

$$\cos x = \sin a \sin b = \tfrac{1}{2}[\cos(a-b) - \cos(a+b)]$$

(Wolf, *Astronomische Mittheilungen*, xxxii, p. 387). Longomontanus considered it preferable to logarithms in their then state. Wittich has left no writings and Buergi's references are in manuscript; the earliest published statement of the method is by Raymarus Ursus Ditmarsus in his *Fundamentum Astronomicum*, 1588, f°16^2, who gives the rules of the method, dedicating the diagram of the first case, from which its demonstration may be gathered, to Paul Wittich of Breslau.

It seems clear that some report of this method was brought to Napier about 1590 by John Craig, physician to James VI of Scotland, who up to this date enjoyed Tycho's friendship. See Glaisher's article in the *Encyclopædia Britannica* (12th edition), p. 872, second column, referring to a visit of Craig to Napier and the latter's communication to him of the method of logarithms, where an extract from a letter of Kepler's is also given, to the effect: '... Scotus quidam literis ad Tychonen 1594 scriptis jam spem fecit Canonis illius Mirifici.' It is, however, very doubtful if this 'Scotus quidam' was Craig, as a controversy broke out between Tycho and a John Craig in 1590 on the subject of the distance of comets. The circumstances remain obscure, for the evidence of Tycho's friendship with Craig is an inscription of date 1588 in a book (in the possession of the University of Edinburgh), the date upon the title-page of which is 1610; yet they are evidence that Napier was engaged upon logarithms at least twenty years before their publication.

The books exhibited are:

47. Nicolai Raymari Vrsi Dithmarsi. | Fundamentum Astronomicum: | Id Est. | Nova Doctrina | Sinvvm Et Triangvlorum. | ... | Cui adiunctae sunt: | ... | VI. Solutis plerorumq́ue Triangulorum per solam prosthaphaeresin. | VII. Eiusdem prosthaphaereseos Apodixis, Causa, ac Demonstratio. | ... | Argentorati. | Excudebat Bernhardus Iobin. | 1588.

4°; size $6 \times 7\frac{1}{2}$ inches.

The demonstrations are found on pp. 16^2, 17.

Lent by the University of Edinburgh.

48. Astronomia | Danica | Vigiliis & opera | Christiani S. Longomontani | Professoris Mathematum, in Regia Acad. Hauniensi, | elaborata. . . . |
| Amsterodami | Ex officina Typographia Guiljelmi I. Caesii | Anno M. DC. XXII.

4°; size $7 \times 9\frac{1}{2}$ inches.

The historical note and demonstration of the method of prosthaphæresis are found on pp. 10 *et seqq.*

Lent by the ROYAL OBSERVATORY.

49. Tychonis | Brahe Dani, | . . . Epistolarvm Astro- | nomicarum Libri. | . . . | Imprimebantur Vraniburgi Daniae, | Prostant | Francofurti apud Godefridum Tampachivm | M D CX.

4°; size $6\frac{1}{2} \times 8\frac{1}{2}$ inches.

In Tycho's hand is written: 'Clarissimo & varia excellentiq; eruditione ornatissimo viro D. Doctori Joanni Craijco Edenburgi in Scotia Medecinam facienti, & Mathematico peritisso dono misit Tycho Brahe & hanc scripsit manu sua Uraniburgi anno 1588 November 2.'

Lent by the UNIVERSITY OF EDINBURGH.

IX.—SPECIMENS ILLUSTRATING THE SUBSEQUENT DEVELOPMENT OF LOGARITHMIC TABLES

50. New | Logarithmes. | The | First inuention whereof, was, by the Ho- | nourable Lo: Iohn Nepair Baron | of Merchiston, and Printed at Edinburg in | Scotland, Anno: 1614. In whose vse was | and is required the knowledge of Al- | braicall Addition and Substra- | ction, according to | + and — . . . By IOHN SPEIDALL, professor of the Mathematickes, and are to | bee solde at his dwelling house in the Fields, on the | backe side of Drury Lane, between Princes | streete and the new Play- | house. |

The 10. Impression. | 1628.

First published 1620.

4°; size $5\frac{1}{2} \times 7\frac{1}{2}$ inches. A^2, B–M^4, followed by 16 pages without signature.

There is no introductory matter.

The first part contains tables of the logarithms of sine, tangent and secant in the form

Deg. 30. Nombers for the

M.	Sine.	Comp.	Tangent.	Comp.	Secant.	Comp.	
0	930685	985616	945069	54931	14384	69315	60

(31 lines to the page.)

thus the entries are the complements of Napier's logarithms or are true Napierian logarithms in the modern sense of the word. The pages at the end are logarithms of numbers from 1 to 1000 and the halves of the same arranged thus :—

000000	693147	000000	1	000000	346573	5000000	1
693147		9306853	2	346573		4653427	2

the numbers are in the middle column, the first column to the left of it contains Napier's logarithm, the second the difference and the third the complement of Napier's logarithm ; on the right we have the halves of these respectively.

Lent by the Royal Observatory.

51. Arithmetiqve | Logarithmetiqve, | ov | La Construction & Vsage des Ta- | bles Logarithmetiques. | Par le moyen desquelles, multiplication se | fait par addition, diuision par sous- | traction, l'extraction de la racine quar- | rée par bipartition, & de la racine cu- | bique par tripartition, &c. Finalement la | reigle de trois, & la resolution des trian- | gles tant rectilignes que spheriques par | addition, & soustraction. | Par Edmond Vvingate, Gentil- | homme Anglois. | In tenui, sed non tenuis vsusvè laborvè. |

A Paris, | Chez Melchoir Mondiere, | demeurant en l'Isle du Palais, ruë | de Harlay, aux deux Viperes. | M. DC. XXV. | Auec Priuilege du Roy.

Size $4\frac{1}{4}\times2\frac{1}{4}$ inches, ă, ĕ, ĭ, A–Z, Aa, Bb ; the sheets are alternately 4° and 8°.

The book opens with a dedicatory preface: 'A très-haut et très-puissant Prince, Monsievr, Gaston de France, Frère vniqve dv Roy, Duc d'Anjou,' etc.

In a short preface, chiefly abridged from Briggs's *Arithmetica Logarithmica*, 1624 (No. 27 above), logarithms are defined and explained. Napier's name is not mentioned; but Briggs is referred to as the author of the logarithms to base 10; and in the closing paragraph Gunter's trigonometrical logarithmic tables are duly noted.

The natural numbers 1 to 1000 are given with their corresponding logarithms arranged in double columns to the page. These are taken from Briggs's *Chilias Prima*, 1617 (No. 22 above). There is no mark to distinguish the characteristic from the mantissa, the logarithm of 2 being 03010299, that of 20 being 13010299, and so on. The first differences are given underneath the logarithms. These tables occupy thirty-four pages, by far the greater part of the book being taken up with the trigonometrical tables, giving the logarithms of the sines and tangents for all degrees and minutes of the quadrant. These are taken from Gunter, *Canon Triangulorum*, 1620 (No. 23 above). The following shows the arrangement over two pages facing each other:—

M.	Sin. 24.	Tang. 24.	M.	Sin. 65.	Tang. 65.
0	96093133	96485832	60	99607301	103514168
1	96095969	96489230	59	99606738	103510769
2	96101635	96492627	58	99606175	103507372
3					

This is the first volume of Briggian logarithms published on the Continent.

Lent by JOHN SPENCER, Esq., F.I.A., London.

52. [DENNIS HENRION. Traicté des Logarithms. Paris, 1626] mentioned by Glaisher.

53. Tabulae Logarithmicae, | or | tvvo tables | of Logarithmes: | The first containing the Logarithmes | of all numbers from 1, to

100000 : | Contracted into this portable Volume | By NATHANIEL ROE Pastor of | *Benacre* in SVFFOLKE. |

The other, the Logarithmes of the right | *Sines* and *Tangents* of all the Degrees and Mi- | nutes of the Quadrant, each degree being divided | into 100 Minutes, and the Logarithme of the | Radius or Semidiameter being 10,00000,00000. | Unto which is annexed their admirable use | for the resolution of all the most necessary | Problems in Geometrie | Astronomie, | Geographie, and Navigation. By Edm: Wingate Gent.

London | Printed by M. Flesher for Philemon Stephens | and Christopher Meridith at the Golden-Lion | in Pauls Church-yard. MDCXXXIII.

8°; size $6\frac{5}{8}\times4\frac{1}{4}$ inches.

The logarithms are to 8 figures, including the characteristic, *i.e.* to 7 figures according to present custom. The logarithms of the sines, cosines, tangents, cotangents are to 10 figures.

Lent by UNIVERSITY COLLEGE, LONDON.

54. [Edmund Wingate]; A Logarithmeticall Table, whereby the Logarithme of any number under 400000 may be readily discovered.

Imprinted at London. M. DC. XXXV.

12°; size $2\frac{3}{4}\times3\frac{3}{4}$ inches; A-G.

Contains logarithms (mantissa only) to 6 decimals of numbers from 1 to 10000, with a difference column; and bound in the same volume:

Artificiall Sines and Tangents, or A Logarithmeticall Table, containing the Logarithmes of the Sines and Tangents of all the Degrees & Minutes of the Quadrant, The Logarithme of the Radius, or Semidiameter being put 10,000000.

Contains logarithmic sines, cosines, tangents, cotangents for every minute to 6 places.

Lent by the ROYAL OBSERVATORY.

55. Trigonometria Britanica: or, the Doctrine of Triangles, In Two Books. The first of which sheweth the construction of the Naturall, and Artificiall Sines, Tangents and Secants, and Table

of Logarithms: with their use in the ordinary Questions of Arithmetick, Extraction of Roots, in finding the Increase and Rebate of Money and Annuities, at any Rate or Time propounded. The other, the use or application of the Canon of Artificiall Sines, Tangents and Logarithms, in the most easie and compendious wayes of Resolution of all Triangles, whether Plain or Spherical.

The one composed, the other Translated, from the Latine Copie written by Henry Gellibrand, Sometime Professor of Astronomy in Gresham-College, London.

A Table of Logarithms to 100,000, thereto annexed, with the Artificial Sines and Tangents, to the hundred part of every Degree; and the three first Degrees to a thousand parts. By John Newton, M.A.

London: Printed by R. & W. Leybourn, and are to be sold by George Hurlock at Magnus Church corner, Joshuah Kirton at the Kings Arms, and Thomas Pierrepont, at the Sun in Pauls Church-yard, and William Fisher at the Postern neer Tower-Hill. MDC. LVIII.

F°; size $11\frac{1}{4}\times7\frac{1}{2}$ inches; A-Z, Aa, Bb.

Contains translation of Gellibrand's *Trigonometria Britannica.*

4°; same size; A-Z.

Contains to 8 decimals logarithms of numbers up to 100,000, with a column showing the logarithm of the difference between successive entries.

4°; same size; Aa-Ll.

Logarithmic sines, cosines, tangents, cotangents to 0°·01 and to 8 decimals, with difference columns; log sin 0° and log tan 0° are given as zero.

4°; same size; A-D.

Logarithmic sines and tangents for the first 3 degrees to 0°·001, with logarithmic differences.

Lent by the ROYAL OBSERVATORY.

56. Organum | Mathematicum | Libris ix. explicatum | à | P. Gaspare Schotto | E Societate Jesu, | . . . .

Herbipoli, | Sumptibus Johannis Andreæ Endteri, & Wolf- | gangi Jun. Haeredum. | Excudebat Jobus Hertz Typographus Herbipol. | Anno M. DC. LXVIII. | Prostant Norimbergae apud dictos Endteros.

4°; size $6\frac{1}{2}\times 8$ inches.

On p. 134 is a calculating machine based upon the method of Napier's rods; the multiples for the ten units are inscribed side by side down the surface of a single cylinder; a number of such cylinders are placed in a box alongside one another and are revolved by handles carried out through the side of the box; the table of products for any number is then exhibited by turning these handles. A plate of such a machine is given, but it is not clear whether it had been constructed or not.

Lent by J. Ritchie Findlay, Esq.

57. [Henry Sherwin]. Mathematical Tables, contrived after a most Comprehensive Method. . . .
London. 1726. (First Edition, 1705.)

Contains a chapter from Wallis's Treatise on Algebra, *Of Logarithms, their Invention and Use*; a paper by E. Halley (*Phil. Trans.*, No. 216), giving the series $q+\frac{1}{2}q^2+\frac{1}{3}q^3+\ldots$, and various transformations of it and application of it to numerical calculations to many places of decimals; some further applications, Abraham Sharp's logarithms of certain numbers to 61 places; the value of π to 72 places; logarithms of numbers from 1 to 100,000 to 7 places of decimals, with differences and proportional parts; natural and logarithmic sines, secants, and tangents, to single minutes and 7 places, with difference columns; the same for versed sines, but without differences, and an appendix on some uses of the preceding tables. Great care was taken in reading these tables, and a list is given of errors discovered in Briggs, Vlacq, and Newton.

Lent by the Royal Observatory.

58. A Mathematical Compendium, . . . Explaining the Logarithms with new Indices; Nepair's Rods or Bones; . . . By Sir Jonas Moore Knight, Late Surveyor General of his Majesties Ordinance. The Fourth Edition.
London. 1705.

A convenient six-figure table of the period.

Lent by J. RITCHIE FINDLAY, Esq.

59. Geometry Improv'd: 1. By a Large and Accurate Table of Segments of Circles, Its construction and various Uses in the Solution of several difficult Problems. With Compendious Tables for finding a true Proportional Part, and their Use in these or any other Tables; exemplify'd in making out Logarithms or Natural Numbers from them to sixty Figures, there being a Table of them for all Primes to 1100, true to 61 Figures. . . . by A(braham) S(harp), Philomath.
London: Printed for Richard Mount on Tower-Hill, and John Sprint in Little-Britain. 1717.

F°; size 7×8 inches.
The series employed for logarithms is

$$\log\frac{z+x}{z-x}=2\frac{x}{z}+\frac{2}{3}\frac{x^3}{z^3}+\ \cdot\ \cdot\ \cdot$$

and the use of the restricted table of primes is illustrated by the example of finding log π (on p. 35), viz. 3·14159, . . . (to 60 places) . . . ×139×229=100000, 03575, . . . (to 58 places), and this lies within his table for interpolation. The logarithms of primes are given on p. [56], . . . namely from 1 to 1097, and by units from 999990 to 1000010, followed by a table of differences up to the tenth difference inclusive, for the latter part of the table.

Lent by the ROYAL OBSERVATORY.

60. The Anti-Logarithmic Canon. Being a Table of Numbers, Consisting of Eleven Places of Figures, corresponding to all Logarithms under 100000. Whereby the Logarithm for any

Number, or the Number for any Logarithm, each under Twelve Places of Figures, are readily found. . . .

To which is prefix'd, An Introduction, Containing a short Account of Logarithms, and of the most considerable Improvements made, since their Invention, in the Manner of constructing them. By James Dodson.

London: Printed for James Dodson, at the Hand and Pen in Warwick-Lane; and John Wilcox, at Virgil's Head, opposite the New Church in the Strand. 1742.

F°; size $8 \times 12\frac{1}{4}$ inches.

The arrangement is the following :—

Logarithms from , 30000 to , 30339. Numbers from 19952623150 to 20108978047.										
Log.	0	1	2	3	4	5				
⋮							......	......	......	......
, 3010	1999·8618696 4604,90	9079187 5,01	9539686 12	0000200 2000 ·.· 22	0460722 33	0921255 43	....	......	......	......

the differences being interlinear, with a note of the change of the first units. At the foot is a table of proportional parts. An appendix follows, on reducing weights and measures to decimal notation.

Lent by the Royal Observatory.

61. Tables of Logarithms, for All Numbers from 1 to 102100, and For the Sines and Tangents to every ten seconds of each degree in the Quadrant; as also, for the Sines of the first 72 minutes to every single second. . . . By William Gardiner.

London: Printed for the Author, in Green Arbour Court, near St Sepulchre's Church, Snow-Hill, by G. Smith, in Stanhope-street, Clare-market. MDCCXLII.

F°; $8\frac{3}{4} \times 10\frac{1}{4}$ inches.

Vlacq was used as a source for these tables, but his numbers were carefully examined. Chiliad 1, 101, 102 are given to 8 figures, the remainder to 7 figures; differences and proportional parts in the margin. The logarithmic table begins with log sines to 7 figures and to single seconds, as far as 1° 12′, followed by log sines, cosines, tangents, cotangents, to intervals of 10″, with columns of differences. A table of Logistical Logarithms follows, that is, for N″ is tabulated log 3600/N, to 4 places. After this is a table to 20 places for numbers up to 1143, and from 101000 to 101139, with first, second, and third differences; finally, a similar table of antilogarithms from ·00000 to ·00139. These tables are beautifully printed.

Lent by the ROYAL OBSERVATORY.

62. Tables de Logarithmes, contenant les Logarithmes des nombres, depuis 1 jusqu'à 102100, & les Logarithmes des Sinus & des Tangentes, de 10 en 10 secondes, pour chaque degré du quart de Cercle, avec différentes autres Tables, publiées ci-devant en Angleterre par Monsieur Gardiner. Nouvelle Edition, Augmentée des Logarithmes des Sinus & Tangentes, pour chaque seconde des quatres premiers degrés.

A Avignon, Chez J. Aubert, Imprimeur-Libraire, rue de l'Epicerie. M.DCC.LXX.

F°; 12×9 inches.

Gardiner's form and beautiful printing are reproduced, with the extension of the logarithms of sines and tangents to single seconds as far as 4°, and the addition at the end of a table of hyperbolic logarithms from 1·00 by steps of ·01 up to 10·00, and for 100·0, 1000, 10000, 100000, to 7 figures.

Lent by the ROYAL OBSERVATORY.

63. Tavole Logarithmiche del Signor Gardiner corrette da molti Errori Occorsi nelle Edizioni Inglese e Francese . . . Edizione Prima Italiana.

In Firenze. MDCCLXXXII.

4°; $8\frac{1}{4} \times 5\frac{1}{2}$ inches.

The format and arrangement are somewhat changed. The 20-figure logarithms are given, at the beginning, for the first chiliad only, which makes them of very little use. Logarithms of numbers are given to 7 figures up to 100000, and to 8 figures up to 100200, and thence for 10021 to 10800. Single seconds are given of sines and tangents for the first 4 degrees, thereafter by intervals of 10 seconds for sines, cosines, tangents, cotangents. A table of hyperbolic logarithms follows from 1 to 130 to 20 places, and for 131 to 1000 to 7 places. At the end is a table of proportional parts.

Lent by the ROYAL OBSERVATORY.

64. Tables Portatives de Logarithmes, contenant les Logarithmes des Nombres depuis 1 jusqu'à 108000, les Logarithmes des Sinus et Tangentes, de seconde on seconde pour les cinq premiers degrés, de dix en dix secondes pour tous les degrés du quart de cercle, et, suivant la nouvelle division centésimale, de dix millieme en dix millieme. . . . Par Francois Callet.

A Paris, chez Firmin Didot, Imprimeur du Roi, de l'Institut, et de la Marine, Libraire pour les Mathémat., &c. rue Jacob, n° 24, 1795.

Stereotyped Edition. First Edition, 1783.

8°; size $8\frac{1}{2} \times 5\frac{1}{2}$ inches.

Contains numerous valuable tables of ordinary and hyperbolic logarithms to many places of decimals, tables for conversion from the one to the other, besides the contents indicated above.

Lent by the ROYAL OBSERVATORY.

65. Table of Logarithms of Sines and Tangents for every Second of the First Five Degrees, and of the Sines, Cosines, Tangents and Cotangents for every Ten Seconds of the Quadrant. By F. Callet.

Paris, 1795—(Tirage 1827).
Stereotyped and Printed by Firmin Didot, Rue Jacob, No. 24.

This is an extract from Callet's Tables above, printed upon yellow-tinted paper for Babbage.

Lent by the Royal Observatory.

66. Johann Carl Schulze, Wirklichen Mitgliedes der Königl. Preussischen Academie der Wissenschaften Neue und Erweiterte Sammlung Logarithmischer, Trigonometrischer und Anderer zum Bebrauch der Mathematik unentbehrlicher Tafeln. 2. Bände.

Berlin, 1778. Bei August Mylius, Buchhändler in der Brüderstrasse.

The title and preface also in French.

4°; size $8\frac{1}{2} \times 5$ inches.

Besides Briggian logarithms it contains Wolfram's hyperbolic logarithms to 48 places of decimals, from 1 to 2200, and thereafter for primes up to 10009, followed by trigonometrical tables to 7 figures, and others.

Lent by the Royal Observatory.

67. Tables of Logarithms of All Numbers, from 1 to 101000; and of the Sines and Tangents to Every Second of the Quadrant. By Michael Taylor, Author of the Sexagesimal Table. . . .

London: Printed by Christopher Buckton, Great Pulteney Street; and Sold by Francis Wingrave, Bookseller, Successor to Mr. Nourse in the Strand. M.DCC.XCII.

F°; size 13×11 inches. There is a preface by Neville Maskelyne, Astronomer Royal, who informs us that Taylor interpolated these tables from Vlacq, and then reduced them to 7 places, and also gives particulars of the pains taken to secure accuracy. Taylor was a computer to the Nautical Almanac. The tables are very well printed.

Lent by the Royal Observatory.

68. Tables des Logarithmes des Sinus et Co-Sinus, Calculés de Seconde en Seconde, pour tous les Degrés du Quart de Cercle, Par M. L'Abbé Robert, Curé de Ste Geneviéve de Toul.
Première Partie. [1784.]
Labor improbus omnia vincit. Virg. [Georg. I. 145] . . . Tangentes et Co-Tangentes. . . .
Second Partie.

Size $9\frac{1}{2} \times 15$ inches. Two large manuscript volumes. No account of their calculation is given, nor does Lalande give any (*Bibliographie Astronomique*, p. 688) when he mentions their reception in 1784. The calculations are to 7 figures. Presumably they were interpolated from Vlacq. The volumes belonged in succession to Lalande, Delambre, Babbage, and Lord Crawford, and contain some MS. notes: 'Compared this MS. with eleven errors of Callet's tables (1814), mentioned in Schumacher, and also in Ferassac's Journal, June 1825, and find only two which are corrected.—C. B.'

In 1828 they were compared with Taylor's Tables (No. 67 above) by the Board of Longitude. Nineteen errors were detected in Taylor, and the Errata published by Pond, as a fly-sheet, in 1830. Pond mentions that the arrangement for the examination was made by Dr Young. This fly-sheet is pasted into the present volume.

Lent by the ROYAL OBSERVATORY.

69. Thesaurus Logarithmorum Completus, ex Arithmetica Logarithmica, et ex Trigonometria Artificialis Adriani Vlacci collectus. . . . A Georgio Vega. . . . Lipsiae in Libraria Weidmannia. 1794.

Title also in German.

6°; size 13×8 inches.

Contains logarithms of numbers from 1 to 100999 to 10 figures, with full differences, log sines, cosines, tangents, cotangents to 10 figures, for every second up to 2°, thereafter for every 10 seconds with differences, length of the arc for single degrees to 11 figures, Wolfram's natural logarithms to 48 places, for numbers from 1 to 2200, and thereafter for primes up to 10009.

Lent by the ROYAL OBSERVATORY.

70. Neue trigonometrische Tafeln für die Decimaleintheilung des Quadranten berechnet von Johann Philipp Hobert Professor der Mathematik und Physik an der Königl. Preussischen Militärakademie des Artilleriekorps und Ludewig Ideler Astronom der Königl. Preussischen Akademie der Wissenschaften.
Berlin, im Verlage der Realschulbuchhandlung. 1799.

4°; size $8\frac{1}{4} \times 5$ inches.

The tables contain natural and also logarithmic sines, cosines, tangents and cotangents to 7 places of decimals, by intervals of 0,00001=10 centesimal seconds=3·24 sexagesimal seconds, followed by several other decimal tables, and a short table of logarithms of numbers.

Lent by the ROYAL OBSERVATORY.

71. Tables Trigonométriques Décimales, ou Tables des Logarithmes des Sinus, Sécantes et Tangentes, suivant la Division du Quart de Cercle en 100 Degrés, Du Degré en 100 Minutes, et de la Minute en 100 Secondes; Précédées de la Table des Logarithmes des Nombres depuis Dix Mille jusqu'à Cent Mille, et de plusieurs Tables Subsidiaires: Calculées par Ch. Borda, revues, augmentées et publiées par J. B. J. Delambre, Membre de l'Institut national de France et du Bureau des Longitudes.
A Paris, de l'Imprimerie de la République An IX. (*i.e.* 1801).

4°; size $9\frac{1}{2} \times 7$ inches.

The tables were published by Delambre after the death of Borda. There are two prefaces, one by Borda and the second by Delambre. The chief table of logarithms of numbers is to 7 figures, but in addition they are given to 11 figures from 1 to 1000, and from 100000 to 102000; then follows a trigonometrical table to 11 figures, for every 10 (centesimal) seconds, up to 10 minutes, and thence for every 10 minutes up to 50 degrees; and to the same number of figures, hyperbolic logarithms for numbers from 1 to 1000. The main trigonometrical table runs to 7 decimals, going by intervals of 1″, centesimal, from 10′ up to 3°, and thereafter by intervals of 10″.

Lent by the Royal Observatory.

72. Nouvelles Tables Astronomiques et Hydrographiques, contenant . . . une Nouvelle Table des Logarithmes, des Sinus, Cosinus, Tangentes et Cotangentes, de Seconde en Seconde, Pour les Quatre-Vingt-Dix Degrés du Quart du Cercle. Par V. Bagay, Professeur d'Hydrographie.

Paris. Firmin Didot Père et Fils, Imprimeurs du Roi, de l'Institut et de la Marine, Libraires pour les Mathématiques, etc. Rue Jacob, No. 24. 1829.

4°; size 10×8 inches.

This well-known work contains many useful astronomical and other tables, besides logarithms of numbers to 7 places from 0 to 21599—log 0 is curiously given as zero—along with what is probably the best and most convenient of the older logarithmic tables of trigonometrical functions to single seconds.

Lent by the Royal Observatory.

73. Logarithmic Tables to Seven Places of Decimals containing Logarithms of Numbers from 1 to 120,000, Numbers to Logarithms from ·0 to 1·00000, Logarithmic Sines and Tangents to Every Second of the Circle, with Arguments in Space and Time, and New Astronomical and Geodesical Tables. By Robert Shortrede, F.R.A.S., &c.

Edinburgh: Adam and Charles Black, . . . Andrew Shortrede, Printer. MDCCCXLIV.

4°; size 10×6½ inches.

By far the greater part of this book of logarithmic tables was edited by Edward Sang, as appears from two letters (now in the possession of the Royal Society of Edinburgh) written to Sang by Lieutenant Shortrede, who was at the time in India. The special feature of the volume is the table of logarithmic sines and tangents to every second of the circle; the type is clean, but the numbers are somewhat crowded on the page.

Lent by the ROYAL OBSERVATORY.

74. Seven-Figure Logarithms of Numbers from 1 to 108000 and of Sines, Cosines, Tangents, Cotangents to every 10 seconds of the Quadrant, with a Table of Proportional Parts, by Dr. Ludwig Schrön, Director of the Observatory of Jena, &c., &c. Fifth Edition, corrected and stereotyped. With a Description of the Tables added by A. de Morgan, Professor of Mathematics in University College, London.

Williams and Norgate, 14 Henrietta Street, Covent Garden, London, and 20 South Frederick Street, Edinburgh.

Frederick Vieweg and Son, Brunswick. 1865.

8°; size 10×7 inches.

This is the best of the earlier seven-figure tables in respect to type, paper, arrangement, and general care in production.

Lent by the ROYAL OBSERVATORY.

75. Tables of Logarithms, By Charles Babbage.
London. 1831.

Twenty-one volumes of experiments with various inks and papers, and twenty-eight complete sets of the tables on different coloured papers. These volumes are experiments made by Babbage to find the least trying combinations of colours of paper and ink for continuous use by a computer.

Lent by the ROYAL OBSERVATORY.

76. A New Table of Seven-Place Logarithms of All Numbers from 20000 to 200000, by Edward Sang, F.R.S.E., Secretary of the Royal Scottish Society of Arts. . . .

London: Charles and Edwin Layton . . . Fleet Street. 1871.

8°; 7×10 inches.

By carrying the table from 20000 to 200000 in place of from 10000 to 100000, interpolation for the logarithm of any number between 1 and 2 is reduced 10-fold; the whole calculation was remade from the beginning, the logarithms of primes up to 2027 being calculated to 28 places, and those of the products of these primes being retained to 15 places.

Lent by the Royal Observatory.

77. Logarithmic, Trigonometrical, and Astronomical Tables: Forty-seven quarto volumes in manuscript (1848 to 1890). By Edward Sang, LL.D.

These volumes were gifted in 1907 to the British nation by the Misses Sang, daughters of the late Dr Edward Sang, the President and Council of the Royal Society of Edinburgh being appointed custodiers with powers to publish such parts as may be judged useful to the scientific world.

Thirty-four of the volumes have to do with the logarithms of numbers. Volumes 1 to 3 contain the details of the steps of the calculations on which the results contained in the next thirty volumes are based. Volume 4 contains the logarithms to 28 places of the primes consecutively to 10,037, with occasional ones beyond. Volumes 5 and 6 contain the logarithms to 28 places of all composite numbers up to 20,000. From these the succeeding twenty-seven volumes are constructed giving the logarithms to 15 places of all numbers from 100,000 to 370,000.

The remaining volumes contain natural sines, and logarithmic sines and tangents, to 15 places, the angles being expressed in the centesimal division of the quadrant; and tables of mean anomalies and circular segments.

These last are of peculiar interest. If we write the equation for Kepler's problem of finding the mean anomaly (m) of a planet's orbit from the eccentric anomaly (u) in the form

$$\begin{aligned} m &= u - e \sin u = u - \cos \psi \sin u \\ &= \tfrac{1}{2}\{(u+\psi) - \sin (u+\psi)\} + \tfrac{1}{2}\{(u-\psi) - \sin (u-\psi)\}, \end{aligned}$$

we see that m may be derived from a knowledge of u and ψ ($=\cos^{-1}e$) by a double reference to a table which gives the quantity $\frac{1}{2}(x-\sin x)$ with argument x. This quantity is the half segment of angle $2x$ in a circle of unit radius. With this design Sang first calculated a table of single entry of circular segments, from which the mean anomalies could be derived by a double reference, and later a table of double entry, with arguments u, e, from which it could be derived by a single reference, encumbered, however, by troublesome interpolations. An account of his method was published in the *Memoirs of the Academy of Turin*, 1879, p. 187.

Lent by the ROYAL SOCIETY OF EDINBURGH.

78. Nouvelles Tables Trigonométriques Fondamentales, contenant les logarithmes des lignes trigonométriques de centième en centième du quadrant avec dix-sept décimales, et de dix en dix secondes avec quatorze décimales, Par H. Andoyer Professeur à la Faculté des Sciences de l'Université de Paris Membre du Bureau des Longitudes . . .

Paris: Librairie Scientifique A. Hermann et Fils, 6, Rue de la Sorbonne, 6. 1911.

4°; size $8\frac{1}{2}\times 11\frac{1}{2}$ inches.

These tables are the result of a computation entirely new, resting chiefly upon the formulæ:

$$\cos\frac{\pi}{2}x=\left(1-x^2\right)\left(1-\frac{x^2}{3^2}\right)\left(1-\frac{x^2}{5^2}\right)\ .\ .\ .$$

$$\sin\frac{\pi}{2}x=\frac{\pi}{2}x\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{4^2}\right)\left(1-\frac{x^2}{6^2}\right)\ .\ .\ .$$

An interesting introduction concludes with some results of scrutiny of other tables; it appears that there are ten cases in which Briggs's results in the *Trigonometria Britannica* are in error by thirteen units and upwards, in the 14th place, while the tenth decimal of Vlacq's *Trigonometria Artificialis* and of Vega's *Thesaurus* has no claim to reliability, being frequently in error by six units. The tables proper are sexagesimal, the centesimal calculations serving as control.

Professor Andoyer has contributed to the present volume an interesting account of his method of calculation. His paper follows this bibliography.

Lent by the AUTHOR and by the ROYAL OBSERVATORY.

79. Logarithmic Trigonometrical Tables with Eight Decimal Places containing the Logarithms of All Numbers from 1 to 200000, and the Logarithms of the Trigonometrical Functions for Every Sexagesimal Second of the Quadrant. . . . Newly Calculated and Published by Dr J. Bauschinger, University Professor and Director of the Imperial Observatory of Strassburg, and Dr J. Peters, Professor Assistant of the Royal Astronomical Calculating Institute of Berlin.

Leipzig. Published by Wilhelm Engleman, 1910.

8º; size $7\frac{1}{2}\times10\frac{1}{2}$ inches.

First Volume. Tables of Logarithms to Eight Places of All Numbers from 1 to 200000.

Contains the Preface to both volumes. The greatest part of the work was done by a specially constructed calculating and typing machine which built the functions up from their second differences. The authors directed their attention to rendering the eighth place certain and to avoiding errors due to interpolation from the old ten figure and other tables. A list of some errors found in Briggs and Gellibrand is given. In order to avoid large differences the logarithms of numbers are given for every unit from 100000 to 200000, thence from 20000 to 100000. In the former part no differences exceed 3 figures, in the second from 20000 to 43000 they run to 4 figures.

Second Volume. Table of the Logarithms to Eight Places of the Trigonometrical Functions for every Sexagesimal Second of the Quadrant, 1911.

Besides the trigonometrical functions, the functions

$S = \log \sin x - \log x$, $T = \log \tan x - \log x$ are tabulated for each second of the first 5 degrees.

80. Logarithmic Table to Seven Places of Decimals of the Trigonometrical Functions for Every Second of the Quadrant, elaborated by Professor J. Peters, Observator of the Royal Astronomical Calculating Institute of Berlin.

Stereotype Edition.

Published by Wilhelm Engleman, Leipzig, 1911.

This is reduced to seven figures from vol. ii of the preceding work, with the addition of the proportional parts from 1° 20′ onwards.

81. Natural Sines to every second of arc and eight places of decimals. Computed by Mrs E. Gifford. Manchester: Printed by Abel Heywood & Son. 1914.

8º; $9\frac{1}{2} \times 6\frac{1}{2}$ inches.

The sines to 10″ were copied from the *Opus Palatinum* of Rheticus (No. 41 above). The sines to 1″ were interpolated by the Thomas calculating machine from Rheticus's figures for 10″, each being copied to 10 places and obvious mistakes corrected so that the differences ran in descending series.

For fuller details as to method of construction, see Mrs Gifford's own account, read at the Tercentenary Congress and printed in this volume.

Lent by Mrs Gifford.

82. Tracts on Mathematical and Philosophical Subjects . . . By Charles Hutton, LL.D. and F.R.S., &c.

London. 1812.

Tract xx (vol. i, pp. 306-340), 'History of Logarithms.'

Tract xxi (vol. i, pp. 340-454), 'The Construction of Logarithms.'

This work was shown for reference.

Lent by the Royal Observatory.

INDEX TO THE BIBLIOGRAPHY

Roman numerals refer to the sections.

Indian numerals refer to the articles (1 *to* 82), *or to the pages* (177 *to* 240).

A. Mainly Authors and Books.

B. Printers and Publishers.

FUNDAMENTAL TRIGONOMETRICAL AND LOGARITHMIC TABLES

H. ANDOYER, Professor of Astronomy in the University of Paris

(Translated by the Editor, Dr C. G. KNOTT.)

Assembled to-day, under the auspices of the Royal Society of Edinburgh, to celebrate the third centenary of the invention of logarithms by John Napier, we are permeated by a common feeling of admiration and gratitude for this illustrious man, who equipped science with the most powerful of weapons for overcoming the difficulties of calculation; and we all wish in turn to bring to his memory a just tribute of homage.

Now the memory of great men may be honoured in many ways. We may, above all, devote ourselves to the reconstruction of the evolution of their thought with help from the too few relative documents which remain to us; or we may praise their work and study with care its radiation through the world and down the ages: nevertheless, it may still be well to seek to develop and perfect this work.

This last is the track along which I have been working for several years past, thanks to the generous concurrence of the University of Paris, which I have the very great honour to represent at this Congress. I will ask permission to engage your attention for some minutes with an account of what I have already done, and of what I hope still to do in continuation of the work of Napier. Perhaps it would have been better to avoid so personal a subject; you will pardon me, however, by attaching no undue weight to this consideration, because I desire greatly to have your advice as a help towards the completion of what has been begun; and how could I ever receive advice more authoritative?

The invention of logarithms is intimately associated by Napier himself with trigonometry, since his table gives at once the values of the natural sines, borrowed from Reinhold, and their *artificial* values or logarithms, for every minute in the sexagesimal division of the degree. We have good reason therefore to call to mind at the same time the various fundamental tables, both trigonometrical and logarithmic, which we have inherited from the mathematicians of the end of the sixteenth and the beginning of the seventeenth centuries: these may be compared to an inexhaustible mine which still continues to be worked.

In the first place comes the *Opus Palatinum de Triangulis*, commenced by Georgius Joachimus Rheticus, and finished by L. Valentinus Otho (Neustadt, 1596). This work, now inaccessible, save in certain libraries, contains to ten places the natural values of the six trigonometrical functions for every ten seconds sexagesimal.

Immediately beside it must be placed the *Thesaurus Mathematicus, sive Canon Sinuum* by Bartholomæus Pitiscus (Francfort, 1613); as rare as the *Opus Palatinum*, it contains the natural values of sines and cosines to fifteen places for every ten seconds.

Then in 1624 there appears in London the *Arithmetica Logarithmica* of Henry Briggs, in which we find the logarithms to fourteen places of numbers from 1 to 20,000 and from 90,000 to 100,000. It is also a work difficult of access, as are likewise the following.

Under the same title of *Arithmetica Logarithmica*, Adrian Vlacq publishes at Gouda, in 1628, the logarithms to ten places of the first 100,000 numbers; then the *Trigono-*

metria Artificialis of the same author (Gouda, 1633) gives to ten places also the logarithms of the trigonometrical functions for every ten seconds.

Finally, the *Trigonometria Britannica* of Henry Briggs, published by Henri Gellibrand (Gouda, 1633), contains the natural values and the logarithms of the six trigonometrical functions for every hundredth of the sexagesimal degree, that is, for every 36″. The logarithms of the sines are to fourteen places, their natural values to fifteen, while those of the tangents and secants are reduced to six.

Such is the work of the founders themselves. Some idea of the labour involved may be gained when we recall that these great calculators had not at their disposal the fruitful resources of the Infinitesimal Analysis.

We should therefore admire them unreservedly, even though they have not attained perfection. We are bound to state, in fact, that the valuable works which have just been enumerated abound in inaccuracies. Purely typographical errors being discounted, the irregular march of the differences often leads us quickly to recognise that we cannot in general be sure of the last figure; and more than once we meet with graver faults, particularly in the beginning of the first edition of the *Opus Palatinum*, before the corrections had been applied by Pitiscus to the cotangents and cosecants of the small degrees.

Without stopping longer to detail these errors, more or less frequent and more or less grave according to the authors, I hasten to say that from the practical point of view they are only of slight importance, since we require very rarely so large a number of decimals.

I am surprised, however, that G. Vega simply re-edited the two volumes of Vlacq without trying to correct the last decimal. His *Thesaurus Logarithmorum Completus* (Leipzig, 1794) has been, however, much used (for ten decimals are not always superfluous), and if the logarithms of the numbers are in general exact to that extent, we cannot say as much of the trigonometrical part. The third decimal of the logarithm of the sine of 45° is in error ! and we find errors which amount to as much as six units in the last figure.

The reproduction of the *Thesaurus* by aid of photozincography by the Geographical Institute of Florence has rendered the greatest service; but it has not made the errors of the original edition disappear.

M. Max de Leber published at Vienna in 1897 a list of errata of the tables of Vega: it is, however, very incomplete.

A great effort was made in France towards the end of the eighteenth century to construct fundamental tables *de novo*. The *Tables du Cadastre*, calculated under the direction of G. Riche de Prony (from 1794 to 1799), contain in particular the logarithms to 10 decimals of numbers from 1 to 200,000 ; the natural sines to 22 decimals for every centesimal minute ; the logarithms to 12 decimals of the sines and tangents for each hundred thousandth of the quadrant. But they have remained in manuscript: the Geographical Service of the Army in Paris published in 1891 an edition reduced to 8 decimals.

The same fate has till now befallen the tables calculated by E. Sang, of which until the present Congress I could speak only in terms of the short account given in the Proceedings of the Royal Society of Edinburgh. The manuscript volumes are now on view at the Royal Society and partly in duplicate in the Exhibition. Sang calculated, without being able to publish, the logarithms to 15 decimals of numbers up to 370,000,[1] and a table of natural sines, of which I do not know the extent.[2]

[1] *Note by Editor.*—These tables occupy 27 manuscript volumes, the foundation of the whole being the logarithms to 28 figures of the prime numbers up to 10,037, with occasional ones beyond, and in two other volumes the logarithms to 28 figures of all numbers to 20,000, excepting the higher primes whose logarithms had not been calculated.

[2] *Note by Editor.*—The sines of arcs differing by the 2000th part of the quadrant were calculated to 33 places, and from these was constructed a table of sines, to 15 decimals, with first and second differences, of arcs differing by 1′, centesimal division, that is, the 10,000th part of a quadrant. From these were constructed also tables of log sines and log tangents to 15 places, the former with first, second and third differences, the latter with first differences.

To terminate this account, already too long, of original researches I ought to refer to numerous abridged tables. I shall cite only that published in 1911 at Berlin by M. J. Peters, where we find the natural values of sines and cosines to 21 decimals for every 10 seconds sexagesimal for all the quadrant, and for every second for the first six degrees.

From what precedes we may conclude that the work of the founders has not been surpassed in extent during these 300 years, and that it remains disfigured by inherent errors. There is, therefore, a strong scientific motive for taking it up again and carrying it to a higher degree of perfection. This motive is not purely speculative, as might be supposed *a priori*: the welcome accorded to the reproduction of the *Thesaurus* of Vega proves it abundantly.

The insufficiency of the usual tables to seven decimals for precise calculations in geodesy and modern astronomy has been long recognised; and Professors Bauschinger and Peters have rendered a great service in publishing their valuable tables to eight decimals. But to obtain these correct they could not content themselves with the *Thesaurus*, but were compelled to institute a direct interpolation from the *Trigonometria Britannica*.

It might perhaps be supposed that this work of revision is quite unnecessary as regards the natural values of the trigonometrical ratios. On the contrary, these values, abandoned after the invention of logarithms, are coming each day into more general use, since they are better fitted for the growing practice of calculating directly by means of machines without the intermediary of logarithms; and already much use has been made of the table of sines and cosines to seven decimals extracted by M. W. Jordan from the *Opus Palatinum*.

The new Fundamental Trigonometrical and Logarithmic Tables which are now proposed are not intended to be used continuously for calculations, but only to serve as a truly solid base for every ulterior publication adapted to different practical needs, and also to make possible the performance without excessive labour of calculations of exceptional precision. They should therefore be sufficiently extended, but not so as to make their extension an obstacle to their publication and use.

How I have been led to undertake the construction and publication of such tables, and how I propose to bring to a satisfactory completion the work already begun, I now ask your permission to explain.

From my youth a natural taste has made me familiar with logarithmic literature, and I have been haunted with the desire to investigate for myself the manner in which a mathematical table may be constructed. During the summer vacation of 1908, and rather for the sake of pure work than under the influence of any precise design, I amused myself—that is the proper word—by taking up the calculation of the formulæ due to Euler, which gave the developments in series of $\log \cos \frac{\pi}{2}x$ and $\log \sin \frac{\pi}{2}x$ (correct to 21 decimals), also of the $\log \tan \frac{\pi}{4}(1-x)$ (correct to 18 decimals).

I have been able to point out some inaccuracies, in general not very grave, in several works which use these formulæ.

These calculations, and all those of which I have still to speak, were made anew entirely by myself, with no other help than Crelle's valuable Multiplication Tables, recalculating even the fundamental values of π and of the modulus M as well as their logarithms. They have also been verified in many ways. I have, moreover, constantly followed the general rule to admit no result which has not been rigorously verified and to grudge no time spent in effectively carrying out the verifications.

With the formulæ obtained I have calculated directly the logarithms of the trigonometrical functions for each hundredth of the quadrant, making use of the relations which give tang a and sin $2a$ as functions of sin a and cos a in such a way as to reduce notably the mass of calculations.

No very definite reason can be assigned for the choice of this interval. It may be justified on the ground that these first documents may serve equally well as the immediate base for tables constructed according to the sexagesimal or centesimal division. But the truth is that I wished only at the time to control a certain table which I had at hand and in which the same results were given to 15 decimals.

Having the table of values of $\log \cos \frac{\pi x}{200}$ and of $\log \sin \frac{\pi x}{200}$ for each whole value of x from 0 to 50, I formed the successive differences of these numbers, and deduced from them the *variations* of the corresponding functions for unit value of x, where by variations of the different orders of $f(x)$ I mean the coefficients of the different powers of h in Taylor's series for $f(x+h)$. These calculations, carried to 18 decimals so as to make sure of 17, furnish a first fundamental table which allows, to the same approximation, the solution of all the problems of logarithmic trigonometry.

To omit nothing essential I ought to say that, as regards the manner of writing the last figure, I have abandoned the ordinary rule, and even the indication (of doubtful utility) which in well-known tables, such as those of Schrön, serves to show if the last figure is forced or not. It appeared to me preferable to adopt uniformly the rule of T. N. Thiele, which consists in forcing the last figure only if the following number is greater than or equal to 750.., but also to place some characteristic indication (I have chosen a small +) after the last figure when this same following number is comprised between 250.. and 750.. In this manner, each sign + being counted as a half unit of the last retained decimal, the error made does not exceed in absolute value the quarter of this unit, and the ordinary calculations are carried out with remarkable simplicity, the whole acquiring thereby a superior precision. It should be clearly understood also that as regards the result of a calculation rounded off according to this rule when suppressing one or several figures, the indication of the last retained decimal cannot be regarded as certain; but if we have a superior limit of the possible error we deduce from it easily the limit of error which may result from the mode of marking adopted, and, in all cases, this superior limit is less by at least a quarter of a unit of the last decimal than that which would result from the application of the usual rule.

Not till I had obtained the logarithms of the trigonometrical functions and their variations for each hundredth of the quadrant did the idea occur to me to draw up from them a complete list of errata of the trigonometrical part of the *Thesaurus* of Vega, the inaccuracy of which has for long distressed me. Then encouraged by the advice of Henri Poincaré and of G. Darboux, and assured of the help of the University of Paris for the publication of my future work, I was very quickly led to enlarge my programme and to undertake definitely the direct construction of a new and complete table to fourteen decimals of the logarithms of the trigonometrical functions. This table appeared at Paris in 1911, a large quarto volume of 600 pages.

I have adopted the uniform interval of ten seconds sexagesimal; it is, in fact, the only one which gave me a sufficient extension without leading me to excessive calculations and to insurmountable difficulties of publication. If, however, the decimal division of the circle and not of the quarter of the circle, proposed formerly by Briggs, had prevailed, it would have been almost equivalent to have chosen as interval the 100,000th part of the circumference, which is 12·96 seconds sexagesimal.

I now give the methods which, after several trials, I have definitely adopted in order to gain the object aimed at. A simple interpolation enables me at the very start to extend the table of the function log cos x to all values of x which are multiples of 9 minutes, up to 45°, since the hundredth of the quadrant is equal to six times 9 minutes. At the same time I was able to calculate without trouble the variations of the same function for an increment of ten seconds, but for every 18 minutes only, that interval being sufficient for the sequel. In the values of the function I have retained fifteen decimals; in those of the successive variations up to the seventh order I have retained respectively 17, 19, 22, 24, 27, 29, and 32 decimals.

I then formed a quite similar table, but extending only from 0° to 3°, for the function

$$S(x) = \log \frac{\sin x}{x''},$$

x'' being the measure of the arc x in seconds.

The fundamental table of the logarithms of cosines for every 10″ from 0° to 45° was now easily constructed by the well-known method of differences. Let α be an angle, a multiple of 18′; we easily deduce, from the variations of $\log\cos\alpha$ for 10″, the differences of different orders which ought to be inscribed on the same line as $\log\cos\alpha$ or else below and above. We thus obtain two columns which commence with the angle α and which we extend above and below to the angles $\alpha \pm g'$, so that each of them contains 54 lines. To construct these tables we write the initial values of the function and of its successive differences to 15, 17, 19, 21, 23, . . . decimals, until we are able to form *a priori* a last column composed of numbers which may be regarded as constant; in fact, it is in this case the column of fourth or fifth differences. The method I have just described is that used by Prony in the calculation of the great Cadastre tables. It is open to criticisms which have been formulated by Sang; but it is quite easy to modify it suitably so as to secure a perfect exactitude. This is a point of which I early recognised the necessity before I was acquainted with Sang's criticisms.

In Prony's method the last difference is regarded as rigorously constant throughout the whole interval covered by one column; moreover, the transition from one difference to the preceding is effected by successive additions or subtractions of numbers whose last two figures are suppressed, rounding them off purely and simply according to the usual rule. In this manner the two columns which ought to agree on one and the same line, corresponding in our case to an odd multiple of 9′, are often considerably out of agreement, and that is a grave disadvantage. This I think I have succeeded in remedying in the following fashion:—In the first place, the last difference used is not taken constant, but it is written sometimes in defect and sometimes in excess, account being taken of any change that may have occurred in its variation, and in such a manner that the extreme values of the preceding difference, which is easily calculated ahead, are found quite exact. A little familiarity and some precautions suffice to realise this arrangement in a satisfactory manner. In the second place, in the successive additions or subtractions which are necessary in passing from one difference to the preceding, I have carefully taken account of the accumulated effect of the neglected decimals, exactly as if I had really added or subtracted them. Thanks to these precautions the different columns are in perfect agreement along the whole extent of their last lines, the divergences being always insignificant. And as the extreme values of the logarithmic cosines had already been obtained, we could still better appreciate the degree of exactitude attained.

The table of the logarithmic sines is deduced from that of the logarithmic cosines by simple application of the formula $\sin \alpha = \sin 2\alpha / 2 \cos \alpha$; but instead of being applied directly to the functions themselves, save for values of α which are multiples of 9′, it has been applied to their first differences. In this manner we are certain to diminish the systematic errors, and above all to avoid every accidental error, since each of the numbers which we are led to write in using the same arrangement as for the logarithmic cosines contributes to the formation of the logarithmic sine which ends each column and which is known in advance.

Applying this procedure to the table of logarithmic cosines, and putting $\sin 2a$ equal to $\cos(90°-2a)$, we obtain the logarithmic sines of angles to 22° 30′; a second application, profiting from the preceding results, gives the logarithmic sines from 22° 30′ to 11° 15′; and so on. It is at once seen that as 0° is approached an accumulation of errors must of necessity take place and finally become sensible, since each of the parts of the table of logarithmic sines is a condensation within a region more and more contracted of the whole already obtained.

To meet this inconvenience and at the same time to test once more the accuracy of the calculations from 0° to 3°, the logarithmic sines have also been calculated directly starting from the differences of the function $S(x)$, reduced at first to tabular form just like the logarithmic cosines. The comparison of the results was found to be quite satisfactory.

The table of logarithmic tangents follows immediately from the preceding; as above, and for the same reasons, the first differences are formed to begin with. Finally, to facilitate calculations relative to the small angles, the function $T(x)$ analogous to $S(x)$ and equal to $\log \frac{\tan x}{x''}$ was put in tabular form, but only from 0° to 3°.

All these calculations were completed by the month of March 1910. They required only a little attention and method, and worked out more quickly than might at first have been expected.

Thanks to the precautions taken, all of which I have not been able to detail, the only errors possible in the original manuscript are errors which should compensate themselves so as to be inappreciable: their probability is extremely small.

The printing required a little more than one year. The proofs were corrected by myself with the greatest care, by the method of eye and ear; that is to say, an assistant read the proof aloud, while I followed on the original manuscript. For greater safety the same process was repeated on the final sheets, and in this way I have been able to make a list of errata which contains only insignificant corrections, and the special object of which was to remedy defects arising from pulling.

Before I had finished the calculation of my table of the logarithms of the trigonometrical functions, the idea occurred to me that it would be advantageous to do the same for their natural values. This new work would, in truth, be much more considerable than the first, since among these natural values there are no relations so simple from the point of view of calculation as those which hold among their logarithms. In fact, it was only by the end of March 1914 that I was able to finish my new manuscript begun in June 1910.

These new tables, which have been printing since the month of April, will form a large quarto volume of about 1000 pages, divided into three parts.

Naturally, I have followed the same plan as for the table of the logarithmic functions; I have preserved the same general arrangement; I have constantly followed the same principles and have observed the same precautions; I may even add that the experience already acquired has allowed me to secure still greater accuracy. I shall now limit myself to a rapid indication of the different stages passed through in succession. Starting afresh, I repeated the calculation, to twenty-four decimals, of the formulæ which give the development in series of the six trigonometrical functions of the arc $\frac{\pi x}{2}$.

Beginning with these formulæ, I obtained without trouble, exact to twenty decimals, the values of the functions

$$\sin \frac{\pi x}{200}, \cos \frac{\pi x}{200}, \tan \frac{\pi x}{200}, \cot \frac{\pi x}{200}, \sec \frac{\pi x}{200}, \operatorname{cosec} \frac{\pi x}{200},$$

$$g(x)=\frac{200}{\pi x} - \cot \frac{\pi x}{200}, \ h(x)=\operatorname{cosec}\frac{\pi x}{200} - \frac{200}{\pi x},$$

for each integral value of x from 0 to 50. To simplify the calculations, I made use of some relations connecting these functions, and especially of the two formulæ

$$\operatorname{cosec} 2\alpha=\tfrac{1}{2}(\cot \alpha+\tan \alpha)$$

$$\cot 2\alpha=\tfrac{1}{2}(\cot \alpha-\tan \alpha).$$

I formed at the same time the variations per unit of x of the functions considered, except of course for the cotangent and cosecant, which have in the neighbourhood of $x=0$, $x=0$ as pole, and consequently vary too rapidly; for this purpose I followed a purely numerical method, starting simply from the differences.

This first fundamental table was then interpolated so as to give me the values of the six trigonometrical functions for every nine minutes up to 45°, and at the same time the functions

$$G(x)=\frac{1}{x} - \cot x, \ H(x)=\operatorname{cosec} x - \frac{1}{x}$$

for every nine minutes also, up to 15°. Moreover, I calculated, per increment of ten seconds and for every eighteen minutes, the variations of the functions $\sin x$, $\cos x$, and $\sec x$ as far as 30°, and of the function $H(x)$ as far as 18°. Similarly, I calculated the variations of $\tan x$ per increment of five seconds, and for every nine minutes up to 30°. The values of the functions are given exact to seventeen decimals, and the successive variations to the corresponding approximation, that is to say, to 20, 22, . . . decimals.

All these preliminary calculations offer no difficulty, but they are long and numerous and have consumed much of my time.

The table of sines and cosines up to 30° was constructed directly by means of the preceding data; to complete it up to 45° I made use of the relations

$$\sin(30°+\alpha)=\cos\alpha-\sin(30°-\alpha)$$
$$\cos(30°+\alpha)=\cos(30°-\alpha)-\sin\alpha;$$

it being clearly understood that, as formerly and as in all analogous cases which follow, these formulæ have not been applied directly to the functions themselves but to their first differences.

The final results are given to 15 decimals, and similarly for all the other trigonometrical functions.

The table of secants as far as 30° was constructed directly; it required more time, since for this function it was necessary to take account of the sixth differences. For the table of tangents I was obliged, in view of what follows, to take an interval a half less, that of five seconds, and it is for this reason that the corresponding variations were in the first place calculated for every nine minutes and per increment of five seconds; nevertheless, the first differences only had to be obtained for every five seconds, and the tangents themselves were calculated for every ten seconds. In this way I worked directly up to 30°. Finally, the table of the function $H(x)$ was also calculated directly as far as 15° for every ten seconds.

To complete to 45° the tables of the secants and tangents, and to set up those of the cotangents and cosecants as well as that of the function $G(x)$ as far as 15°, it sufficed to apply the formulæ already cited which connect the values of $\tan x$ and $\cot x$ with those of $\cot 2x$ and $\operatorname{cosec} 2x$.

Moreover, in order to avoid the accumulation of errors which would follow a too frequent repetition of the application of these formulæ, I calculated also the values of $\cot x$ and $\operatorname{cosec} x$ up to 15° with the help of the tables of $G(x)$ and $H(x)$, using an auxiliary table constructed directly and giving the values of $\frac{1}{x}$.

The entirely satisfactory comparison of the results thus obtained by two absolutely different methods for the co-tangents and cosecants up to 15° has fully verified the whole of the calculations relating to the functions other than the sine and cosine.

To facilitate the work when the angles are small, the table of the functions $G(x)$ and $H(x)$ is joined to the others; it corresponds to that of the functions S and T in the logarithmic table, but it is necessary to carry it as far as 15°. This is also one of the many reasons which have prolonged the work.

As already stated, the printing of these new tables has begun; I hope to have it finished in two years.

To complete the work of reproduction of the fundamental tables nothing more is needed than to construct a new table of the logarithms of numbers. It is a work to which I have already given some thought, and the following seems to me to be the most suitable plan to follow.

It is necessary at first to lay down the principle that such a table should not contain the logarithms of more than 100,000 numbers. This is a range which could not be practically exceeded. We may therefore with advantage choose the numbers from 100,000 to 200,000; for it is always easy by a very simple multiplication or division by a number less than ten to change any number whatever into a number commencing with the figure 1. Moreover, if the table proposed is to be used only for calculations to ten decimals (an approximation which is sometimes necessary and, except under very special circumstances, always sufficient), it would be quite useless to take account of second differences for purposes of interpolation. Finally, it should be noted that if we wished then to find the logarithms of numbers from 20,000 to 100,000, nothing would be simpler; we have merely to apply as many times as would be necessary the elementary formula

$$\log a = \log 2a - \log 2;$$

and this formula would be applied directly to the first differences as in all the analogous cases already met with.

This is how I propose to realise the plan. It is very easy, in the first place, to determine by rapidly converging series the modulus M and the logarithms of some fundamental numbers. We may then calculate directly, by simple divisions, the variations per increment of unity of the function log x for each even number x comprised between 1000 and 2000, and deduce from them the values of the function. Nothing is simpler then than to construct the proposed table by the method of differences, following exactly the same method as for the trigonometrical tables, the only change being that the columns which will be formed will be of a hundred lines, and no longer of 54 only. It would be necessary then to take precautions still more strict, into the details of which it is useless to enter, in order to avoid the accumulation of errors. But I have already assured myself that by limiting ourselves to the consideration of the fifth differences we should be able to guarantee the accuracy of the sixteenth and even of the seventeenth decimal.

I have every hope and expectation of publishing this third and last table some years hence; but I do not believe that I could alone complete the task of carrying out all the necessary calculations.

Addition and Subtraction Logarithms are of the greatest service, especially in calculations to more than five or six decimals. It would be therefore very desirable to have the elements necessary for compiling new tables of these logarithms to eight, nine, or even ten decimals. It appears to me that we could attain this very easily, while preserving the arrangement of Leonelli and of Gauss and adopting only another choice of argument.

Using, in fact, the well-known notations of Gauss, and making $B=\log x$, we have

$$A=\log\frac{1}{x-1},\ C=\log\frac{x}{x-1}=A+B,$$

and x varies merely from 1 to 2.

If then we take B instead of A as argument, its variation will be between the limits 0 and 0·30103, and if

$$D=\log\frac{2\sinh\frac{B}{2M}}{B}$$

then

$$A=-\frac{B}{2}-\log B-D$$

$$C=+\frac{B}{2}-\log B-D.$$

For the immediate calculation of A and C, it suffices then to reduce the function D to tabular form, which is extremely simple. It will be easy then to draw up a table of A and of C, adopting the arrangement the more appropriate to the use for which the table is designed. I believe that ultimately it will be found convenient to add this fairly short table of the function D to the fundamental table of the logarithms of numbers.

It appears to me further that to facilitate interpolation in the case of the table of the logarithms of numbers, at least when we do not wish to exceed the approximation of ten decimals, the following method should be adopted :—

Let a be a number of the table (between 100,000 and 200,000), and let it be required to calculate $\log(a+\alpha)$, the number α being between 0 and 1. We must add to $\log a$ the quantity $\log\left(1+\frac{\alpha}{a}\right)$. If then we make $A=\log a-\log\alpha$, the quantity to be added to $\log a$ is precisely the number B which corresponds to A, in the notation of Gauss. Further, this quantity cannot exceed the greatest difference of the table, say 0·000043430; it would be convenient then to give to a suitable approximation the values of A which correspond to the values of B smaller than this limit; and in this manner the interpolation would be immediate, requiring no more trouble than that of finding the logarithm of α to five decimals at the most, and to form A=log a−log α. The construction of this table of interpolation would obviously take very little time.

It is for you, gentlemen, to tell me in all sincerity if it is expedient that I should persevere in these plans and that I should realise these barely sketched projects; it will be the best way of pardoning me for having for so long trespassed on your kind attention.

EDWARD SANG AND HIS LOGARITHMIC CALCULATIONS

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Towards the close of the preceding article Professor Andoyer refers to the necessity for a new fundamental table of logarithms of numbers. His suggestion to limit the tabulation to the logarithms of numbers between 100,000 and 200,000 is one which may at once be acted upon if sufficient encouragement is given from the mathematical world. For already the necessary computation has been more than done by Dr Edward Sang, whose manuscript tables of logarithms to fifteen places of numbers from 100,000 to 370,000, gifted some years ago by his daughters to the British nation, are now in the custody of the Royal Society of Edinburgh. These logarithms are tabulated with the first and second differences, so that the accuracy attainable by their use far exceeds the requirements of the most precise calculations in geodesy, astronomy, or life insurance.

These forty-seven manuscript volumes of logarithms, sines, logarithmic sines and tangents and other astronomical and trigonometrical tables were exhibited in July 1914 at the Napier Tercentenary Exhibition; and all who looked carefully into them must have been greatly impressed with the power of calculation and tenacity of purpose which enabled their author to leave behind such a mass of accurate figuring.

Edward Sang was in many respects a remarkable man. He showed an aptitude for mathematical computation at a very early age, and while still a mere youth began to make original contributions to scientific literature. Any real problem in mathematics, in physics, or in engineering strongly attracted him; and the solutions he gave were of characteristic quality. The main facts of his life are simply stated. He was born in Kirkcaldy on January 30, 1805, was a pupil there under Edward Irving until 1818, when he went to Edinburgh, entered the University, and studied for several years under Professors Wallace and Sir John Leslie. With the exception of two years (1841-1843) spent in Manchester New College as Professor of Mechanical Science, and of twelve years (1843-1854) spent in Turkey in various important academic posts, Edward Sang lived and worked in Edinburgh as a private teacher of mathematics. Here he died December 23, 1890.

A sympathetic account of his life will be found in the obituary notice prepared by Mr D. Bruce Peebles for the Royal Society of Edinburgh, and published in the *Proceedings*, vol. xxi, 1897. Sang's published works include *A New General Theory of the Teeth of Wheels* (1852), two very original treatises on *Arithmetic, Elementary* and *Higher* (1856 and 1857), a *Treatise on the Valuation of Life Contingencies* (1864), and a beautifully printed *Table of Seven-Place Logarithms of all numbers up to* 200,000 (1878), based on his own calculations, to be referred to hereafter. To the *Proceedings* and *Transactions* of the Royal Society of Edinburgh Dr Sang made many contributions. He also communicated a large number of scientific papers and notes to the *Proceedings* of the Royal Scottish Society of Arts, of which he was Secretary for nearly thirty years.

Two very early papers of his are worthy of special mention. In March 1836 he read before the Royal Scottish Society of Arts a 'Suggestion of a New Experiment which would demonstrate the Rotation of the Earth.' His own description is in these words :

'While using Troughton's Top, an idea occurred to me that a similar principle might be applied to the exhibition of the rotation of the earth. Conceive a large flat wheel, poised on several axes, all passing through its centre of gravity, and whose axis of motion is coincident with its principal axis of permanent rotation, to be put in very rapid motion. The direction of its axis would then remain unchanged. But, the directions of all surrounding objects varying on account of the motion of the earth, it would result, that the axis of the revolving wheel would appear to move slowly.'

The instrument designed by Sang to demonstrate the phenomenon so clearly described by him was never constructed, mainly through lack of means ; and it was not till eighteen years later that Foucault, helped by the funds of Imperial France, first effected the demonstration. A gyroscope made by Sang himself is among the exhibits in the Royal Scottish Museum.

On February 20, 1837, Edward Sang read before the Royal Society of Edinburgh an 'Investigation of the Action of Nicol's Polarising Eye-piece.' The paper for some unaccountable reason was not published at the time, nor is it mentioned in the published volume of the *Proceedings*. In the Minute Book of the Society it is, however, recorded as having been communicated. Shortly before his death Dr Sang told Professor Tait that he regarded this unpublished paper as one of his best contributions to scientific literature, for he believed it to be the first complete account of the true theory of the Nicol Prism. After a search the manuscript was found, and was printed in volume xviii of the *Proceedings* (1891), with an explanatory note by Professor Tait. It is not too much to say that this paper, from its intrinsic merit, would have formed one of the most important contributions of its time. In addition to working out the mathematical theory of the Nicol Prism, Sang made the suggestion to construct the polariser of two glass prisms separated by a thin layer of Iceland spar. This suggestion was subsequently made by M. E. Bertrand forty-seven years later (*Comptes Rendus*, xcix, p. 538, 1884).

Edward Sang's work in connection with logarithmic tables began in 1836, when he undertook the editing of Shortrede's *Logarithmic Tables to Seven Places of Decimals*. As we learn from letters received in 1838 from Shortrede, then resident in India, Sang introduced various improvements. Even before that date, however, he had been considering the possibility of making a new set of calculations; but the obvious labour involved prevented him taking any definite action until the year 1848, when he was resident in Constantinople. Having secured a copy of Burckhardt's *Table des Diviseurs*, Sang at once saw how effectively this table of factors could be used in greatly reducing the labour of calculating logarithms. He was further encouraged in carrying out this great plan as it formed an important step in his calculation of the trigonometrical tables with the quadrant divided centesimally and decimally. In his paper 'On the Need for Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor' (*Proc. R. S. E.*, vol. xii, 1884), for which he was awarded the Makdougall-Brisbane Prize by the Council of the Royal Society of Edinburgh, Sang gives a short account of his method of calculation in constructing the various tables he has left in manuscript. These include a table of natural sines calculated to 25 places for each two thousandth part of the quadrant, and to 15 places for each ten thousandth part, the computation having been effected by use of the second differences, verified at short intervals; also tables of logarithmic sines and tangents, the former with first, second and third differences, the latter with first differences only. Another interesting series of tables are those which have to do with his method of circular segments in calculating the mean anomaly. This was first described in a paper published in 1879 in the *Memoirs* of the Turin Academy, and will be found discussed above (page 237) in Professor Sampson's Bibliography of Books exhibited at the Napier Tercentenary Exhibition.[1] Sang's belief that the centesimal division of the quadrant would in time displace the sexagesimal division seems to be less likely than ever of realisation. In any case, however, the change could not be effected either rapidly or completely; and to make his tables immediately useful Sang compiled convenient tables for passing from one system to the other.

Fortunately this question of centesimal as against sexagesimal division does not affect the logarithmic tables of numbers. Beginning independently of all previous work, but testing his results by comparison where possible with earlier calculations, Sang computed the logarithms to 28 places of all primes up to 10,037 and occasional ones beyond, each prime being put in relation to, at least, three others. The greatest discrepancy found was a unit in the 27th place. By combination of these primes the logarithms to 28 places of all integral composite numbers from 1 to 20,000 were calculated and tabulated, the gaps due to the uncalculated primes being comparatively few. From this table there was then constructed by interpolation the great table of logarithms to 15 places of all integral numbers from 100,000 to 370,000. The intention was to have carried the tabulation on to 1,000,000; but time and strength did not permit this continuation. In these calculations Dr Sang was assisted by his daughters in the purely arithmetical part of building up the successive logarithms within given limits by addition of the second differences. He considered that

[1] See also the *Handbook to the Napier Tercentenary Exhibition*, p. 42.

the residual errors necessarily accompanying this method of interpolation could not exceed 3 units in the 15th place. By comparison with simple multiples of the fundamentally calculated primes I have made a large number of tests up to 300,000, and have found the error generally inappreciable in the 15th place, sometimes as much as 1 unit, and very occasionally as great as 2. We may assume with Sang that these logarithms are absolutely accurate to the 14th place.

Here, then, we have in hand the material desiderated by Professor Andoyer as a fundamental basis for all future tabulations of logarithms. There are two ways in which Sang's tables may be utilised for this end. They may be published as they stand with the logarithms to 15 places with first and second differences; or they may be abridged to 12 places with first differences only. If the latter method were adopted, the type might be set up from the original manuscript, but it would not be an easy piece of work even for an expert compositor. It would be safer to make a manuscript copy at the hand of a careful copyist accustomed to figuring. The setting up would then be comparatively simple. In any case the proofs would be compared with the originals, at least twice, by different readers thoroughly experienced in that kind of work. Although the second differences affect the 11th and 12th place, varying from 43 to 11 throughout the range 100,000 to 200,000, the variation is so slow that the second differences need not be entered more than once at the foot of each column of fifty logarithms. The effect of the second difference in interpolation could be assigned almost at sight, even in the inverse case of interpolating the number for a given logarithm not contained in the table. By arranging the logarithms in columns of fifty with the first differences in parallel column, we could print on each page 150 numbers with their logarithms, first differences and second differences; and the whole hundred thousand numbers could be printed in one large quarto volume of 667 pages.

It would, however, be in some respects simpler and certainly infinitely more accurate to reproduce Sang's original manuscript pages as line engravings by photography. This was indeed the method which first suggested itself to Dr Burgess and myself when Dr Sang's manuscript volumes were consigned to the care of the Royal Society of Edinburgh. And various considerations have recently made some of us regard with increasing favour this method of utilising to the full the fundamentally important parts of Sang's manuscript volumes.

It should be noted in the first place that the manuscripts have been prepared with extreme care, the figures being beautifully written and entered with great neatness in appropriate spaces in specially ruled paper. The rulings are differently spaced according to the nature of the table. For example, in the Table of the logarithms of integer numbers from 1 to 10,000, there are twenty-five numbers on each page with their logarithms to 28 figures, covering, therefore, 400 manuscript pages in all. This, which forms the basis of the whole, could be reproduced by a slight reduction in 100 pages of 100 numbers to the page.

The logarithms to 15 figures of numbers from 100,000 to 200,000 are also arranged in the manuscript volumes twenty-five to the page, and contain the first and second differences entered in specially prepared ruled paper. Here, also, by a similar reduction four of the manuscript pages could be reproduced as one page containing 100 numbers with their logarithms and first and second differences. The whole could be reproduced in a large quarto volume of 1000 pages.

It should be noted that the final tabulations were made by Dr Sang after three verifications, so that it is almost impossible for any error of tabulation to exist, the only possible errors being those which result from the method of interpolation, and which, as already stated, rarely reach three units in the fifteenth figure.

It would thus be possible to reproduce with an absolute accuracy the manuscript tables giving the 28-figure logarithms of the first 10,000 numbers and the 15-figure tables of numbers from 100,000 to 200,000 conveniently arranged in one volume of 1100 pages. The *Auxiliary Table*, for convenient interpolation when more than 11 or 12 figures were being used, would occupy another hundred pages. In this way the most important part of Sang's logarithmic calculations would be made accessible to the whole mathematical world; and the work of calculating anew a fundamental table of logarithms need never again be undertaken. Such a table would form the source of all future tabulations of logarithms.

It has been estimated that the cost of reproducing by photography these tables would be about one-third or one-fourth the cost of setting them up in type in the usual way. Much time and risk of error in copying, in setting up, and in proof-correcting would be saved; and as regards the accuracy of the two methods there is of course no comparison.

Such, then, is the proposal which I desire to lay before the many interested in logarithms; and what more fitting outcome of the Napier Tercentenary could there be than making accessible to the civilised world the fundamental part of these great tables, calculated by Edward Sang in the very city where John Napier invented the logarithm and gained undying fame as a benefactor of his race?

FORMULÆ AND SCHEME OF CALCULATION FOR THE DEVELOPMENT OF A FUNCTION OF TWO VARIABLES IN SPHERICAL HARMONICS

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(Translated by the Editor, Dr C. G. Knott)

When a function has been expressed as a series of spherical harmonics with constant coefficients, the determination of these coefficients from given values of the function is in the general case one of the most comprehensive operations which can be set before the calculator.

Since Gauss first carried out these operations in a calculation of this kind,[1] efforts have not been wanting to simplify them and make their frequent application possible. The most successful of all in this respect was Franz Neumann,[2] who showed that by a suitable choice of the argument the calculation could be materially shortened.

For the application of Neumann's method H. Seeliger [3] arranged the constant coefficients in tables, and thereby made the calculations so easy and so obvious that even a non-scientific calculator can carry it out. I would now show that some further steps may be taken in this direction, with the advantage that in addition to a further shortening of the calculation the whole process can be carried out by one operation on the calculating machine, since only sums of products have to be formed.

[1] Burckhardt, *Oszillierende Funktionen*, pp. 384 ff.

[2] *Astronomische Nachrichten*, Bd. xvi, p. 313 (1838).

[3] *Sitzungsberichte der Konig. bayer. Akademie der Wissenschaften München: Math.-phys. Classe*, Band xx, p. 499 (1891).

[The given values[1] of the function $f(\mu, \phi)$, where $\cos^{-1}\mu(=\theta)$ is the polar distance and ϕ the longitude, are supposed distributed over a spherical surface, such as the earth's, and the function itself is expressed as

$$f(\mu, \phi)=\sum_{\circ}^{p}\mathrm{Y}^{n}$$

where Y^n is the spherical surface harmonic of degree n and is of the general form

$$\mathrm{Y}^{n}(\mu, \phi)=\sum_{i=0}^{i=n}(\mathrm{A}_{ni}\cos i\phi+\mathrm{B}_{ni}\sin i\phi)\mathrm{P}_{ni}(\mu),$$

where A and B are the $2n+1$ arbitrary constants to be determined, and where

$$\mathrm{P}_{ni}(\mu)=\sqrt{1-\mu^{2}}\frac{\partial^{i}\mathrm{P}^{n}(\mu)}{\partial\mu^{i}},$$

P^n being the zonal spherical harmonic of degree n of one variable.

In Franz Neumann's method $2p(p+1)$ values of the function are taken, namely,

$$f(\mu_1, 0), f\left(\mu_1, \frac{\pi}{p}\right), \ldots . f\left(\mu_1, (2p-1)\frac{\pi}{p}\right)$$

$$f(\mu_2, 0), f\left(\mu_2, \frac{\pi}{p}\right), \ldots . f\left(\mu_2, (2p-1)\frac{\pi}{p}\right)$$

. . . .

. . . .

$$f(\mu_{p+1}, 0), f\left(\mu_{p+1}, \frac{\pi}{p}\right), \ldots . f\left(\mu_{p+1}, (2p-1)\frac{\pi}{p}\right)$$

where $\mu_1 \ldots \mu_{p+1}$ are the $(p+1)$ roots of the equation

$$\mathrm{P}^{p+1}(\mu)=0.$$

[1] The part in square brackets has been added by the translator, so as to make the notations immediately intelligible to the reader.

For each root μ_λ the values of C and S are defined by

$$\epsilon_i p C_i(\mu_\lambda) = \sum_{\nu=0}^{\nu=2p-1} f\left(\mu_\lambda \,.\, \nu\frac{\pi}{p}\right)\cos\nu\frac{i\pi}{p}$$

$$p S_i(\mu_\lambda) = \sum_{\nu=0}^{\nu=2p-1} f\left(\mu_\lambda \,.\, \nu\frac{\pi}{p}\right)\sin\nu\frac{i\pi}{p}$$

where ϵ_i is unity, except when $i=0$ or p, when the value is 2.

If now we write

$$\mathfrak{Z}_{ni}(\mu_\lambda) = \frac{2n+1}{2}\frac{(n-i)\,!}{(n+i)\,!}P_{ni}(\mu_\lambda)a_1$$

$a_1, a_2, a_3, \ldots a_{p+1}$ being given by the solution of the equations

$$a_1\overset{\lambda}{\mu_1} + a_2\overset{\lambda}{\mu_2} + \;\ldots\; + a_{p+1}\overset{\lambda}{\mu}_{p+1} = a_\lambda \quad (\lambda=0, 1, 2, \ldots p)$$

$$\text{where } a_\lambda = \int_{-1}^{+1} x^\lambda dx$$

it may be shown that the constants A and B are determined by the values given in (1) below.]

Take then the expressions for the constants sought, namely,

$$\left.\begin{aligned} A_{ni} &= \sum_{\lambda=1}^{\lambda=p+1} \mathfrak{Z}_{ni}(\mu_\lambda)C_i(\mu_\lambda) \\ B_{ni} &= \sum_{\lambda=1}^{\lambda=p+1} \mathfrak{Z}_{ni}(\mu_\lambda)S_i(\mu_\lambda) \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (1)$$

and combine the known coefficients in the C and S with the $\mathfrak{Z}$, when we obtain

$$\left.\begin{aligned} A_{ni} &= \sum_{\lambda=1}^{\lambda=p+1} \sum_{\nu=0}^{\nu=2p-1} G_{ni}\left(\mu_\lambda,\, \nu\frac{i\pi}{p}\right).f\left(\mu_\lambda,\, \nu\frac{\pi}{p}\right) \\ B_{ni} &= \sum_{\lambda=1}^{\lambda=p+1} \sum_{\nu=0}^{\nu=2p-1} H_{ni}\left(\mu_\lambda,\, \nu\frac{i\pi}{p}\right).f\left(\mu_\lambda,\, \nu\frac{\pi}{p}\right) \end{aligned}\right\} \quad . \; . \quad (2)$$

In every case the $2p(p+1)$ coefficients G_{ni} and H_{ni} (of the latter $p+1$ are *ab initio* equal to zero) are tabulated for each combination n, i, and the operation to be carried out with the calculating machine is then continued quite simply so that each of the given $2p(p+1)$ values of the function

$f\left(\mu_\lambda, \nu\frac{\pi}{p}\right)$ is multiplied with the corresponding G and H respectively, and the sums of all products taken.

After determination of the A_{ni} and B_{ni}, the interpolation formula for the function $f(\mu, \phi)$ becomes

$$\left.\begin{aligned} f(\mu, \phi) = (&P_{00}A_{00} + P_{10}A_{10} + \dots + P_{p0}A_{p0}) \\ &+(P_{11}A_{11} + P_{21}A_{21} + \dots + P_{p1}A_{p1}) \cos\phi \\ &+(P_{11}B_{11} + P_{21}B_{21} + \dots + P_{p1}B_{p1}) \sin\phi \\ &+(P_{22}A_{22} + \dots + P_{p2}A_{p2}) \cos 2\phi \\ &+(P_{22}B_{22} + \dots + P_{p2}B_{p2}) \sin 2\phi \\ &+ \dots\dots\dots \\ &+P_{pp}A_{pp} \cos p\phi \end{aligned}\right\} . \quad (3)$$

where the associated spherical harmonics P are functions of the powers of sin δ and cos δ $(=\mu)$. For convenience of application, the expressions within the brackets in (3) require to be changed into rows which are developed in sines and cosines of multiples of θ; that is, the arrangement takes the form

$$\begin{aligned} f(\theta, \phi) = (&\alpha_{00} + \alpha_{10} \cos\theta + \alpha_{20} \cos 2\theta + \dots + \alpha_{p0} \cos p\theta) \\ &+(\alpha_{11} \sin\theta + \alpha_{21} \sin 2\theta + \dots + \alpha_{p1} \sin p\theta) \cos\theta \\ &+(\beta_{11} \sin\theta + \beta_{21} \sin 2\theta + \dots + \beta_{p1} \sin p\theta) \sin\theta \\ &+(\alpha_{02} + \alpha_{12} \cos\theta + \alpha_{22} \cos 2\theta + \dots + \alpha_{p2} \cos p\theta) \cos 2\phi \\ &+(\beta_{02} + \beta_{12} \cos\theta + \beta_{22} \cos 2\theta + \dots + \beta_{p2} \cos p\theta) \sin 2\phi \\ &+ \dots\dots \end{aligned}$$

The *second* step to be made in preparing once for all for the carrying out of the calculations is that, instead of the above-named tables for the G_{ni} and H_{ni}, similar tables may be immediately constructed for the calculation of the α_{ni} and β_{ni}. This is easily possible, since the α_{ni}, β_{ni} are simple known functions of A_{ni} and B_{ni}.

The α_{ni} and β_{ni} are then obtained as sums of the $2p(p+1)$ products, of which the one factor is $f\left(\mu_\lambda, \nu\frac{\pi}{p}\right)$ and the other factor stands in the table.

Such a table possesses the advantage that a glance enables us to recognise and calculate the influence of a change of a given value of the function $f\left(\mu_\lambda,\ \nu\frac{\pi}{p}\right)$ upon the coefficients α and β; for if $g_{ni\lambda\nu}$ is the value in the table for α_{ni} which corresponds to λ, ν, and $\Delta f_{\lambda\nu}$ the known change, then will $g_{ni\lambda\nu}\Delta f_{\lambda\nu}$ be the corresponding change of α_{ni}, and

$$g_{ni\lambda\nu}\Delta f_{\lambda\nu}\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\cos i\phi\ \begin{cases}i \text{ even}\\ i \text{ odd}\end{cases}$$

the change of the function $f(\theta,\ \phi)$.

In practical work the direct use of the table is not to be recommended, for, although the mode of calculation is indeed very clear, the number of products to be formed is great. It is possible also to supply a much simpler procedure, since the tabulated values of each α_{ni} are for the greater part zero, or equal, or equal and opposite.

We have now left a few of the different coefficients which are to be multiplied by constantly recurring combinations of $f\left(\mu_\lambda,\ \nu\frac{\pi}{p}\right)$. These latter are made up solely out of the sums and differences of the $f\left(\mu_\lambda,\ \nu\frac{\pi}{p}\right)$ without factors, and are quickly formed by calculation with the hand; all further working is best done with the machine.

In practice the calculator will mostly be concerned with developments up to the fourth order of spherical harmonics, and only in exceptional cases will be compelled to go as far as the sixth order. I here restrict myself therefore to the communication of the formulæ and numbers for the case $p=4$; their deduction may be left out, as it is quite simple.

In accordance with the theory, we must take as given values of the function the points of section of the meridian

$$\phi=0°,\ 45°,\ 90°,\ .\ .\ .\ .\ .\ 315°$$

with the parallels whose polar distances are

$$\begin{aligned}\theta_1&=154^\circ\ 58'\ 57''\cdot6\\ \theta_2&=122\ 34\ 46\cdot2\\ \theta_3&=\ 90\ \ 0\ \ 0\cdot0\\ \theta_4&=\ 57\ 25\ 13\cdot8\\ \theta_5&=\ 25\ \ 1\ \ 2\cdot4\end{aligned}$$

forty values in all.

I represent them shortly in the following way :

$$f(\theta_\lambda,\ \nu 45^\circ)=\lambda\nu \qquad \begin{array}{l}\lambda=1,\ 2,\ 3,\ 4,\ 5\\ \nu=0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\end{array}$$

The coefficients α, β are then expressed by the interpolation formula—

$$\begin{aligned}f(\theta,\ \phi)=\ &\alpha_{00}+\alpha_{10}\cos\theta+\alpha_{20}\cos 2\theta+\alpha_{30}\cos 3\theta+\alpha_{40}\cos 4\theta\\ &+(\alpha_{11}\sin\theta+\alpha_{21}\sin 2\theta+\alpha_{31}\sin 3\theta+\alpha_{41}\sin 4\theta)\cos\phi\\ &+(\beta_{11}\sin\theta+\beta_{21}\sin 2\theta+\beta_{31}\sin 3\theta+\beta_{41}\sin 4\theta)\sin\phi\\ &+(\alpha_{02}+\alpha_{12}\cos\theta+\alpha_{22}\cos 2\theta+\alpha_{32}\cos 3\theta+\alpha_{42}\cos 4\theta)\cos 2\phi\\ &+(\beta_{02}+\beta_{12}\cos\theta+\beta_{22}\cos 2\theta+\beta_{32}\cos 3\theta+\beta_{42}\sin 4\theta)\sin 2\phi\\ &+(\alpha_{13}\sin\theta+\alpha_{23}\sin 2\theta+\alpha_{33}\sin 3\theta+\alpha_{43}\sin 4\theta)\cos 3\phi\\ &+(\beta_{13}\sin\theta+\beta_{23}\sin 2\theta+\beta_{33}\sin 3\theta+\beta_{43}\sin 4\theta)\sin 3\phi\\ &+(\alpha_{04}+\alpha_{24}\cos 2\theta+\alpha_{44}\cos 4\theta)\cos 4\theta\end{aligned}$$

In the first place, all the combinations of the given values of the function are to be calculated. They are shown in Table A, being represented by the symbols

$$\begin{array}{l}[1]_1,\ [2]_1,\ [3]_1\ \ldots\ldots\ldots\ldots\ [15]_1\\ [1]_2,\ [2]_2,\ [3]_2\ \ldots\ldots\ldots\ldots\ [15]_2\\ [1]_3,\ [2]_3,\ [3]_3\ \ldots\ldots\ldots\ldots\ [15]_3\end{array}$$

This table gives at the same time an appropriate scheme for carrying out the calculation. The first column contains the forty values of the function arranged in the most convenient order ; the other columns explain themselves. It will be seen that numbers which are to be added or subtracted stand directly under one another. For these simple operations controls are hardly necessary, and are indeed furnished by the mode of their summation. The same process repeats itself constantly so as to become strongly impressed on the memory.

The α and β follow as sums of products, with constant factors, of the numbers just determined. These products are given in Table B. It is there evident that for the finding of the thirty coefficients α, β (six of which are immediately expressible in terms of the others), ninety-two products are necessary. Since these can be immediately formed and summed by means of the calculating machine, a further control other than is furnished in the usual way by the working of the machine is superfluous.

In the second table, for simplification of the numbers the first factors with their tenfold totals are set down ; compensation is effected most simply by dividing the values of the function by ten before using them, whereby as a rule a desirable homogeneity in the whole set of numbers is brought about.

TABLE A.

TABLE A.

Function.	First Sum.	Second Difference.	Successive Sums and Differences.		Successive Differences and Sums.	
10	10 + 14	10 − 14	$(10+14)+(12+16)=a_1$	$(10+14)-(12+16)=b_1$	$(10-14)=c_1$	$(12-16)=d_1$
14						
12	12 + 16	12 − 16				
16						
50	50 + 54	50 − 54	$(50+54)+(52+56)=a_5$	$(50+54)-(52+56)=b_5$	$(50-54)=c_5$	$(52-56)=d_5$
54						
52	52 + 56	52 − 56	$a_1+a_5=m_1$	$b_1+b_5=[1]_1$	$c_1+c_5=[3]_1$	$d_1+d_5=[5]_1$
56			$a_1-a_5=n_1$	$b_1-b_5=[2]_1$	$c_1-c_5=[4]_1$	$d_1-d_5=[6]_1$
11	11 + 15	11 − 15	$(11+15)+(13+17)=e_1$	$(11+15)-(13+17)=f_1$	$(11-15)+(13-17)=g_1$	$(11-15)-(13-17)=h_1$
15						
13	13 + 17	13 − 17				
17						
51	51 + 55	51 − 55	$(51+55)+(53+57)=e_5$	$(51+55)-(53+57)=f_5$	$(51-55)+(53-57)=g_5$	$(51-55)-(53-57)=h_5$
55						
53	53 + 57	53 − 57	$e_1+e_5=p_1$	$f_1+f_5=[7]_1$	$g_1+g_5=[9]_1$	$h_1+h_5=[11]_1$
57			$e_1-e_5=q_1$	$f_1-f_5=[8]_1$	$g_1-g_5=[10]_1$	$h_1-h_5=[12]_1$
			$m_1+p_1=[13]_1$			
			$m_1-p_1=[15]_1$			
			$n_1+q_1=[14]_1$			
20	20 + 24	20 − 24	$(20+24)+(22+26)=a_2$	$(20+24)-(22+26)=b_2$	$(20-24)=c_2$	$(22-26)=d_2$
24						
22	22 + 26	22 − 26				
26						
40	40 + 44	40 − 44	$(40+44)+(42+46)=a_4$	$(40+44)-(42+46)=b_4$	$(40-44)=c_4$	$(42-46)=d_4$
44						
42	42 + 46	42 − 46	$a_2+a_4=m_2$	$(b_2+b_4)=[1]_2$	$(c_2+c_4)=[3]_2$	$d_2+d_4=[5]_2$
46			$a_2-a_4=n_2$	$(b_2-b_4)=[2]_2$	$(c_2-c_4)=[4]_2$	$d_2-d_4=[6]_2$
21	21 + 25	21 − 25	$(21+25)+(23+27)=e_2$	$(21+25)-(23+27)=f_2$	$(21-25)+(23-27)=g_2$	$(21-25)-(23-27)=h_2$
25						
23	23 + 27	23 − 27				
27						
41	41 + 45	41 − 45	$(41+45)+(43+47)=e_4$	$(41+45)-(43+47)=f_4$	$(41-45)+(43-47)=g_4$	$(41-45)-(43-47)=h_4$
45						
43	43 + 47	43 − 47	$e_2+e_4=p_2$	$f_2+f_4=[7]_2$	$g_2+g_4=[9]_2$	$h_2+h_4=[11]_2$
47			$e_2-e_4=q_2$	$f_2-f_4=[8]_2$	$g_2-g_4=[10]_2$	$h_2-h_4=[12]_2$
			$m_2+p_2=[13]_2$			
			$m_2-p_2=[15]_2$			
			$n_2+q_2=[14]_2$			
30	30 + 34	30 − 34	$(30+34)+(32+36)=a_3$	$(30+34)-(32+36)=b_3$	$(30-34)=c_3$	$(32-36)=d_3$
34						
32	32 + 36	32 − 36				
36						
31	31 + 35	31 − 35	$(31+35)+(33+37)=e_3$	$(31+35)-(33+37)=f_3$	$(31-35)+(33-37)=g_3$	$(31-35)-(33-37)=h_3$
35						
33	33 + 37	33 − 37	$a_3+e_3=[13]_3$	$b_3=[1]_3$	$c_3=[3]_3$	$d_3=[5]_3$
37			$a_3-e_3=[15]_3$	$f_3=[7]_3$	$g_3=[9]_3$	$h_3=[10]_3$

TABLE B.

(The first factors are set down with their tenfold totals.)

$$\begin{aligned}\alpha_{00} &= +0{\cdot}3296[13]_1\\ &\quad +0{\cdot}1444[13]_2\\ &\quad +0{\cdot}3021[13]_3\end{aligned}$$

$$\begin{aligned}\alpha_{10} &= -0{\cdot}5973[14]_1\\ &\quad -0{\cdot}1555[14]_2\end{aligned}$$

$$\begin{aligned}\alpha_{20} &= +0{\cdot}5087[13]_1\\ &\quad -0{\cdot}3628[13]_2\\ &\quad -0{\cdot}2917[13]_3\end{aligned}$$

$$\begin{aligned}\alpha_{30} &= -0{\cdot}3246[14]_1\\ &\quad +0{\cdot}5462[14]_2\end{aligned}$$

$$\begin{aligned}\alpha_{40} &= +0{\cdot}1791[13]_1\\ &\quad -0{\cdot}5072[13]_2\\ &\quad +0{\cdot}6562[13]_3\end{aligned}$$

$$\begin{aligned}\alpha_{11} &= +0{\cdot}3155[3]_1+0{\cdot}2231[11]_1\\ &\quad +0{\cdot}8306[3]_2+0{\cdot}5873[11]_2\\ &\quad +0{\cdot}8333[3]_3+0{\cdot}5893[11]_3\end{aligned}
\qquad
\begin{aligned}\beta_{11} &= +0{\cdot}3155[5]_1+0{\cdot}2231[9]_1\\ &\quad +0{\cdot}8306[5]_2+0{\cdot}5873[9]_2\\ &\quad +0{\cdot}8333[5]_3+0{\cdot}5893[9]_3\end{aligned}$$

$$\begin{aligned}\alpha_{21} &= -0{\cdot}6449[4]_1-0{\cdot}4560[12]_1\\ &\quad -0{\cdot}9328[4]_2-0{\cdot}5887[12]_2\end{aligned}
\qquad
\begin{aligned}\beta_{21} &= -0{\cdot}6449[6]_1-0{\cdot}4560[10]_1\\ &\quad -0{\cdot}9328[6]_2-0{\cdot}5887[10]_2\end{aligned}$$

$$\begin{aligned}\alpha_{31} &= +0{\cdot}6382[3]_1+0{\cdot}4513[11]_1\\ &\quad +0{\cdot}3720[3]_2+0{\cdot}2630[11]_2\\ &\quad -1{\cdot}1667[3]_3-0{\cdot}8250[11]_3\end{aligned}
\qquad
\begin{aligned}\beta_{31} &= +0{\cdot}6382[5]_1+0{\cdot}4513[9]_1\\ &\quad +0{\cdot}3720[5]_2+0{\cdot}2630[9]_2\\ &\quad -1{\cdot}1667[5]_3-0{\cdot}8250[9]_3\end{aligned}$$

$$\begin{aligned}\alpha_{41} &= -0{\cdot}7675[4]_1-0{\cdot}5427[12]_1\\ &\quad +0{\cdot}6482[4]_2+0{\cdot}4584[12]_2\end{aligned}
\qquad
\begin{aligned}\beta_{41} &= -0{\cdot}7675[6]_1-0{\cdot}5427[10]_1\\ &\quad +0{\cdot}6482[6]_2+0{\cdot}4584[10]_2\end{aligned}$$

$$\begin{aligned}\alpha_{02} &= +0{\cdot}1823[1]_1\\ &\quad +0{\cdot}6289[1]_2\\ &\quad +0{\cdot}2917[1]_2\end{aligned}
\qquad
\begin{aligned}\beta_{02} &= +0{\cdot}1823[7]_1\\ &\quad +0{\cdot}6289[7]_2\\ &\quad +0{\cdot}2917[7]_3\end{aligned}$$

$$\begin{aligned}\alpha_{12} &= -0{\cdot}1575[2]_1\\ &\quad -0{\cdot}7506[2]_2\end{aligned}
\qquad
\begin{aligned}\beta_{12} &= -0{\cdot}1575[8]_1\\ &\quad -0{\cdot}7506[8]_2\end{aligned}$$

$$\begin{aligned}\alpha_{22} &= +0{\cdot}1272[1]_1\\ &\quad -0{\cdot}0907[1]_2\\ &\quad -1{\cdot}1667[1]_3\end{aligned}
\qquad
\begin{aligned}\beta_{22} &= +0{\cdot}1272[7]_1\\ &\quad -0{\cdot}0907[7]_2\\ &\quad -1{\cdot}1667[7]_3\end{aligned}$$

$$\alpha_{32}=-\alpha_{12} \qquad \beta_{32}=-\beta_{12}$$

$$\alpha_{42}=-(\alpha_{02}+\alpha_{22}) \qquad \beta_{42}=-(\beta_{02}+\beta_{22})$$

$$\begin{aligned}\alpha_{13} &= +0{\cdot}0368[3]_1-0{\cdot}0260[11]_1\\ &\quad +0{\cdot}5872[3]_2-0{\cdot}4152[11]_2\\ &\quad +1{\cdot}1667[3]_3-0{\cdot}8250[11]_3\end{aligned}
\qquad
\begin{aligned}\beta_{13} &= -0{\cdot}0368[5]_1+0{\cdot}0260[9]_1\\ &\quad -0{\cdot}5872[5]_2+0{\cdot}4152[9]_2\\ &\quad -1{\cdot}1667[5]_3+0{\cdot}8250[9]_3\end{aligned}$$

$$\begin{aligned}\alpha_{23} &= -0{\cdot}0997[4]_1+0{\cdot}0706[12]_1\\ &\quad -0{\cdot}9487[4]_2+0{\cdot}6710[12]_2\end{aligned}
\qquad
\begin{aligned}\beta_{23} &= +0{\cdot}0997[6]_1-0{\cdot}0706[10]_1\\ &\quad +0{\cdot}9487[6]_2-0{\cdot}6710[10]_2\end{aligned}$$

$$\alpha_{33}=-\tfrac{1}{3}\alpha_{13} \qquad \alpha_{43}=-\tfrac{1}{2}\alpha_{23} \qquad \beta_{33}=-\tfrac{1}{3}\beta_{13} \qquad \beta_{43}=-\tfrac{1}{2}\beta_{23}$$

$$\begin{aligned}\alpha_{04} &= +0{\cdot}0044[15]_1\\ &\quad +0{\cdot}1392[15]_2\\ &\quad +0{\cdot}3281[15]_3\end{aligned}
\qquad
\alpha_{24}=-\tfrac{4}{3}\cdot\alpha_{04} \qquad \alpha_{44}=+\tfrac{1}{3}\cdot\alpha_{04}$$

NUMERICAL TABLES AND NOMOGRAMS

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(Translated by the Editor, Dr C. G. Knott.)

In a numerical Table of Double Entry arranged in *rows* and *columns*—the common part to a row and to a column being called a *partition*—the partitions of the first row and of the first column serve for the inscription of the respective values of each of the entries, taken according to a natural order; and the result obtained by association of a couple of values of these entries is inscribed in its turn in the partition common to the corresponding row and column.

The classical type of a table of this kind is furnished by the Multiplication Table, or Table of Pythagoras, familiar to all.

To pass from such an aid for calculation to a nomogram, it suffices to bring into correspondence with each value of each of the data not so much the interior of a partition, as a single point marked on one of the borders of the figure. The columns and rows of which we have just spoken are then replaced by simple straight lines parallel to both borders. When the object is to establish a correspondence between each point of one of the borders and the value of the datum to which it refers, the simplest method (although this is by no means necessary) is to take the distance of this point from the origin proportional to this value; in such a manner that if, as is the custom, the values of each set of data increase by equal steps, the corresponding parallels to one of the borders of the figure are equally spaced, and the combination of the two systems of parallels gives to the figure the appearance of a draught board—in Greek ἄβαξ, whence the term *abaque* or abacus.

This arrangement has two advantages as compared with the numerical table. In the first place, every value of the data, outside those which are inscribed, is effectively represented on the figure by the corresponding point of the border whose distance from the origin is expressed, on a chosen scale, by this value, a visual interpolation permitting the representation of all the values of the data within the limits of the figure. In the second place since, under these conditions, there corresponds to each value of the resultant only one point, we may join by a line all points of the plane which correspond to the same value of the resultant, write this value once for all alongside this line so that the resultant as a whole is represented by the same method as serves to represent, by contour lines, the relief of a topographical surface. It is well to remark that when these contour lines are obtained only by simple empiricism, the lines marked by means of the values of the resultant (lines sometimes called *isoplèthes*, but it is evidently simpler to call them merely *côtées*) [1] may be constructed in a precise manner by the methods of analytical geometry.

It was Pouchet who, first in his *Arithmétique linéaire*, published at Rouen in 1795, made a *systematic* use of this mode of representation for functions of two independent variables or, what comes to the same, for equations of three variables; but others before him had thought of making use of the method in particular cases, and we may in this connection cite the interesting Longitude Tables and Horary Tables of Margetts, published in London in 1791.

This mode of representation is evidently applicable to any relation whatever between three variables, but there exists in general, for the third variable, the diagram of lines,

[1] In this translation we shall, however, use the term isopleths or contours.

of which certain points obtained disconnectedly are then joined by a continuous trace. For example, the Table of Pythagoras, translated after this fashion, becomes a diagram of equilateral hyperbolas.

It will evidently be a great advantage, whenever possible, to have to trace only straight lines, or, if need be, circles, when making use of an appropriate geometrical transformation of the 'abaque' already obtained. One such transformation would simply result in the change of the mode of correspondence between points taken on the borders of the diagram and the values of these data. As already remarked, it is not necessary that the distances of the points from the origin should be proportional to the corresponding values of these data. They may be other functions of them, and in certain cases a judicious choice of these functions may transform into straight lines the original isopleths in terms of the third variable.

The first example which was given of such a transformation was that made known in 1842 by Lalanne under the name of *anamorphose logarithmique.* In the simplest case to which it has been applied, that of the product $z_1z_2=z_3$, it consists in bringing the variables z_1 and z_2 into correspondence respectively with the straight lines $x=\log z_1$, $y=\log z_2$, in which case there holds, for the isopleths (z_3), the equation $x+y=\log z_3$. This represents the straight lines perpendicular to the bisectors of the angle between the axes. This principle of anamorphosis has, from the start, displayed great fruitfulness, and has sensibly contributed to the development of representations of this kind. By means of successive generalisations, it has led to the consideration of equations representable not only by two systems of straight lines parallel to the axes of co-ordinates and one other unrestricted system of straight lines, but by three systems of straight lines under no such restrictions. This has been done, independently the one of the other, by Massau and the author of the present note. The latter has also studied equations representable by means of systems of circles.

It is, moreover, very remarkable that the greater part of the equations met with in technical applications belong to a type representable by three systems of straight lines. This circumstance confers a general interest in every improvement of the mode of corresponding representation. The most notable which we may cite is without doubt that which results from the following observation:—On all the nomograms previously described the key of the mode of representation lies in the fact that three isopleths corresponding respectively to each of the variables intersect mutually in one point. These three systems of contours produce a somewhat confused network which at times make the interpolations at sight not very accurate; moreover, when each isopleth is followed from the point of intersection to where the isopleth number is inscribed, there is a risk of passing from this line to a neighbouring one with a consequent error of reading. To ward off completely these inconveniences it suffices, in the case of a nomogram of intersections uniquely composed of straight lines, to have recourse to a dual transformation. This transformation replaces the three systems of isopleths by three systems of marked points between which there is nothing more to be done than take alignments (since three concurrent straight lines are brought by such a transformation into correspondence with three points in a straight line). Such is the idea of the principle which has led to the consideration of collinear-point nomograms. Its application has been rendered practical, thanks to the use of a special dual transformation which will be discussed in the first of the lectures to be given by the author at the Mathematical Colloquium which follows the celebration of the Tercentenary of the invention of logarithms by John Napier.

But it is expedient to add now that the most important advantage offered by the method of collinear points consists in the possibility of admitting doubly infinite systems of points (summits of a network), in virtue of which it has been possible to represent nomographically equations of more than three variables, of which the previous methods gave no convenient representation; such are notably, among many others met with in divers technical applications, the complete equation of the third degree (which enters into various questions of hydraulics and of the resistance of materials), and the fundamental equations of spherical trigonometry which will form the subject of the second lecture to be given by the author at the Colloquium.

ON THE ORIGIN OF MACHINES OF DIRECT MULTIPLICATION

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(Translated by the Editor, Dr C. G. Knott.)

The Handbook of notes relative to the objects exhibited in one of the rooms of the University of Edinburgh on the occasion of the Tercentenary of logarithms is of the highest interest, and will remain in the future a valuable source of instruction on a number of methods and instruments of calculation. But, by reason even of its special object, this Handbook lays no claim to being a complete history of the questions to which its various parts refer. It may then be not without interest to recall in a few words the origin of certain developments realised in the last years of the nineteenth century à propos of calculating machines.

We know that originally such machines, of which the first in date was that of Blaise Pascal (1641), were fit only for effecting simple additions. It was Leibnitz who first conceived the idea of adapting to a machine of this kind a mechanism capable of making it repeat several times in succession, with great rapidity, the addition of one and the same number so as to effect multiplication mechanically. But the theoretical idea of Leibnitz found practical realisation for the first time in the *Arithmomètre*, designed and constructed in 1820 by Thomas de Colmar (without, it is certain, any knowledge of the essays of Leibnitz), which has been perfected since then by himself and others, and which has remained the prototype of machines of additions rapidly repeated so as to yield multiplication.[1]

It was only at the *Exposition Universelle de Paris* in 1887 that there was seen, for the first time, a machine working multiplications directly based on the use of the Table of Pythagoras. This machine was the invention of a man, then quite young, who has since then taken a large share in the development of automobilism, and who, unfortunately, died recently in the strength and prime of life: we refer to Léon Bollée.

There is, in this regard, a curious similarity between the first inventor of machines ' of addition ' and the first inventor of machines ' of direct multiplication.'

Pascal was eighteen years old when he designed his machine, in order to come to the help of his father, whose duty it was as superintendent at this epoch of Haute-Normandie to verify accounts of great length. Similarly Bollée was eighteen years of age when he constructed his machine, in order to assist his father, who was a bell-founder, in carrying out the complicated calculations necessary for determining the model of bells which should sound given harmonies with a certain fundamental tone.

[1] See for fuller details on the history of calculating machines our work, *Le calcul simplifié par les procédés mécaniques et géophysiques*, Gauthier-Villars, 1893; 2nd edition, 1905. We believe that this work gives, for the first time, a general view of calculating machines properly classified.

It is worthy of note that Léon Bollée had not, at this time, any knowledge of any of the pre-existing machines, and it was perhaps due to this happy ignorance that he imagined, at the first effort, arrangements so remarkably original, and notably the calculating plates furnished with tongues of appropriate length which constitute a kind of table of Pythagoras in relief acting directly on the recording apparatus of the machine.

These are the same plates which are found in the machine designed by M. Speiger[1] under the name of *Millionaire*, of which the general arrangement is somewhat simpler than Bollée's machine, bringing it rather under the category of the Arithmometer.

Bollée's machine, as realised by its inventor, was excellent in its action. Its limited circulation in practice is due, on the one hand, to its very high price, and on the other to the fact that Léon Bollée, engrossed by his important work in connection with automobilism, was virtually compelled to lay aside completely the question of calculating machines. We are right in regretting it, for he had still in his drawings divers projects of the highest interest concerning this object, and notably that of a difference machine able to operate to differences of the twenty-seventh order, susceptible, in consequence, of taking the place of the laborious calculations required in all cases in which the method is applied.

[1] See *Handbook of the Exhibition*, p. 119.

A H

NEW TABLE OF NATURAL SINES

MRS E. GIFFORD

Napier published his work on logarithms in 1614. In 1514 was born George Joachim, surnamed Rheticus, whose *Opus Palatinum* has hitherto been the greatest work extant on the properties of the Triangle. It was published in 1596, twenty years after his death, by Valentine Otho under the patronage of the Elector Frederick, hence, I suppose, its name. Had we met in 1896 to celebrate the publication of Rheticus' work, we should all have been using logarithms, and the *Opus Palatinum* would have been of secondary importance. Now we meet, in 1914, to celebrate the publication of Napier's work, we are most of us using calculating machines and natural numbers; and logarithms are fast taking a second place.

I suppose the general use of logarithms had to wait until logarithmic tables of sines and tangents were compiled, just as now the use of calculating machines is hampered by the want of tables of natural numbers. In 1897 Dr Jordan published a table of natural sines which was a reprint of Rheticus to every 10 seconds of arc and 7 places of decimals.

The *Opus Palatinum* itself goes to 10 seconds of arc and 10 places of decimals, an instance of the love of uniformity of the age in which it was published. I do not know how many copies of the *Opus Palatinum* are in existence. I have only seen the one in the Reading Room of the British Museum.

My husband's work in geometrical optics led him to use Jordan's book, but I thought it would be a great saving of time if there were a table to every second of arc with first decimal differences given, so that any sine could be written down directly. As I live some distance from London, I began by copying Jordan's work to 6 places, putting the first four in a column apart, because I found that those four were the same for at least twenty seconds and could be prefixed to the next four figures for any one of those seconds. The next two I put in the next column, the one headed 0. Of course where Jordan's 7th place was a zero I copied to 5 places only for fear the last digit had been raised. Whenever I had the chance of going to London, I copied what I could from the table of Sines in Rheticus. I think there are generally a great many misprints in old books; at any rate there are in Rheticus. Some of these, such as the wrong degree to head the page, the wrong minute, or a wrong figure at the beginning of the sine, are perfectly obvious. But when a misprint occurs later on, say in the 7th or 8th place, only very careful inspection will show it. I collated first of all the differences for every 5 minutes, as in Table A, where it will be seen that the differences decrease in regular order, and that the third differences are identical, or nearly so. Then I collated those for every minute, as in Table B, where the second differences are nearly identical, and where the five first differences add up so as to equal the first of the first differences in Table A. Then for every 10 seconds, as in Table C, to which the same applies.

At first the differences vary so little that the same difference would be correct enough for the whole minute. But though the differences decrease, the difference between the differences increases, and where there was too much of a drop between one 10″ difference and the next, I treated each as an average for the 10″, and took the difference from 0″ to 1″ as higher and from 9″ to 10″ as lower than the one given by Rheticus, using this last only for the middle one, but making all the differences from 0″ to 10″ equal to the one difference given by Rheticus. This is shown in Table D, in which the successive sines are got by addition of the prepared differences.[1] I thought that in calculating from a 10 place sine it would be safe to use 8 places, though when the 9th place is very near 5 this may not be correct enough. I think, by the method in Table D, I might even have got 9 places, but as most machines only go to 8, I left it at that. Table D shows that when the differences are all correct one turn of the handle of the machine gives a new sine. The differences for one second of arc begin with ·00000485, by 30° they have dropped to ·00000420, by 60° to ·00000243 and at 90° they vanish.[2] The difference between 0° and 90° is 1, of which $\frac{1}{2}$ is added by 30° and nearly $\frac{7}{8}$ by 60°. A table of sines has practically the same difference as a table of tangents for the 1st second of arc; but whereas the differences decrease for the sines they increase for the tangents, and by the time one reaches 0° 19′ there is a difference of 1 in the 7th. I have worked out about 11° of natural tangents to every second, again taking Rheticus as the foundation.

Before I met with the *Opus Palatinum* I had worked out about 240 sines, but it meant such enormous labour to get decimal places enough that I gave it up.

Those I had worked were useful for comparison where I suspected a misprint in Rheticus. After I had finished the table, Mr E. M. Nelson kindly lent me a copy of Callet's centesimal table of natural sines. He gives 1000 sines to the quadrant, or one to every 324″, to 15 places of decimals. I compared that with my corrected proofs and found very few errors in mine, none, I think, of more than 1 in the 8th place. Callet's table was particularly useful for comparison, as out of the 1000 sines given between 0° and 90°, only 200 were for even 10″, and therefore coincided with Rheticus, the other 800 being for angles in between. When I was in doubt about the 8th place of any sine, I squared it and the cosine, added

[1] Tables A, B and C are to 10 places, Table D goes to 11.

[2] It is over 78° before there are 6 zeros.

them together and took the number that made the addition come nearest to one.

On page 543 of my Tables will be found a table of sines where the 5th, 6th, 7th, and 8th places are all zeros; and I have tabulated them in a small Table below as far as the eleventh decimal. There are, of course, in addition the well-known exact cases of sine 90° and sin 30°. Of these twenty-eight, 8 have 0 in the 9th place, 2 have 0 in the 4th place and one in the 3rd.

Only one of these 28 sines coincides with Rheticus and none with Callet, so I do not think that such a table has been published before.

There are nearly a million and a half figures in the whole table of sines as it stands; so that, if some errors have been left, I hope any who are good enough to use my table will not be too hard on me.

Angles whose Sines when carried to Eight Places have Zeros in the last Four.

Angle.	Sine.	Angle.	Sine.
89° 11′ 23″	·9999 0000 313	45° 9′ 13″	·7090 0000 463
86 51 41	·9985 0000 025	41 18 27	·6601 0000 238
82 58 42	·9924 9999 498	41 14 20	·6592 0000 180
80 53 42	·9874 0000 092	31 48 35	·5271 0000 160
80 30 18	·9863 0000 074	25 26 9	·4295 0000 406
79 19 37	·9827 0000 095	25 21 35	·4283 0000 085
71 57 9	·9508 0000 207	23 56 6	·4056 9999 698
69 21 28	·9358 0000 302	22 58 2	·3901 9999 819
69 16 36	·9353 0000 266	19 17 35	3303 9999 825
65 15 28	·9081 9999 742	14 53 53	·2570 9999 705
63 19 45	·8936 0000 010	11 31 52	·1998 9999 554
58 3 35	·8485 9999 615	10 42 28	·1858 0000 144
55 50 34	·8275 0000 226	4 0 9	·0697 9999 958
50 21 14	·7700 0000 014	0 25 47	·0074 9999 730

Table showing Method of Interpolation in calculating Sines to every Second of Arc.

	Angle Limits.	First Difference.	Second Difference.	Third Difference.	Angle Limits.	First Difference.	Second Difference.	Third Difference.
A	60°				30°			
	0′ to 5′	·000 726 3042			0′ to 5′	·00 125 90536		
	5′ to 10′	724 4708	18334	17	5′ to 10′	79932	10604	26
	10′ to 15′	722 6357	18351	15	10′ to 15′	69302	10630	26
	15′ to 20′	720 7991	18366		15′ to 20′	58646	10656	
B	60°				30°			
	0′ to 1′	·000 145 4075			0′ to 1′	·000 251 8954		
	1′ to 2′	3342	733		1′ to 2′	8530	424	
	2′ to 3′	2608	734		2′ to 3′	8107	423	
	3′ to 4′	1875	733		3′ to 4′	7684	423	
	4′ to 5′	1142	733		4′ to 5′	7261	423	
		·000 726 3042				·001 259 0536		
C	60° 0′				30 °0′			
	0″ to 10″	·000 024 2396			0″ to 10″	·000 041 9855		
	10″ to 20″	376	20		10″ to 20″	843	12	
	20″ to 30″	356	20		20″ to 30″	832	11	
	30″ to 40″	336	20		30″ to 40″	820	12	
	40″ to 50″	316	20		40″ to 50″	808	12	
	50″ to 60″ or 1′	295	21		50″ to 60″ or 1′	796	12	
		·000 145 4075				·000 251 8954		

	Angle.	Sine.	First Difference.	Angle.	Sine.	First Difference.
D	60° 0′ 0″	·8660 2540380		30° 0′ 0″	·5000 00 0000	
	1″	2782785	242405	1″	41 9860	419860
	2″	3025188	403	2″	83 9719	59
	3″	3267589	401	3″	125 9577	58
	4″	3509988	399	4″	167 9434	57
	5″	3752385	397	5″	209 9290	56
	6″	3994780	395	6″	251 9144	54
	7″	4237173	393	7″	293 8997	53
	8″	4479564	391	8″	335 8849	52
	9″	4721953	389	9″	377 8700	51
	10″	4964340	387	10″	419 8550	50

ERRATA IN MRS GIFFORD'S 'NATURAL SINES'

Page 3, Headline: after Cosine insert 90°.
,, 4, At 0°16′29″ for '8479' read '9479.'
,, 9, At 1°4′50″ for '5814' read '5813.'
,, 9, In difference table under 484·73, opposite '5' for '242·3' read '242·4.'
,, 10, At 1°18′23″ omit dot.
,, 14, At 1°56′40″ for '0039' read '0339.'
,, 57, At 9°0′0″ for '3446' read '3447.'
,, 73, At 11°42′0″ for '8729' read '8730.'
,, 80, At 12°51′21″ omit dot.
,, 180, At Headline: instead of '20°' read '29°.'
,, 394, At 65°15′30″ for '0405' read '0406.'

Initial Dots must be supplied at the following places:—

,, 116, From 18°52′51″ to 18°54′19″.
,, 181, From 29°48′7″ to 29°49′49″.
,, 193, At 32°35′19″.
,, 210, From 34°38′45″ to 34°38′49″.
,, 217, From 35°43′37″ to 35°44′49″.
,, 228, From 37°31′3″ to 37°34′59″.

Note by Editor.—Mrs Gifford's table of 'Natural Sines' to every second of Arc and eight places of decimals may be got by applying to Mrs Gifford, Oaklands, Chard, Somerset.

THE ARRANGEMENT OF MATHEMATICAL TABLES

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The object of this paper is to discuss the following points in regard to the arrangement of Mathematical Tables.

1. *Typography.*
   - (a) Projecting *versus* uniform sized figures.
   - (b) Large or small type.
   - (c) Grouping of columns.
   - (d) The method of indicating a change in a leading figure.
   - (e) The use of coloured paper or coloured ink.
   - (f) Provision of a lateral index.
   - (g) Paper, Printing and Binding.
2. *Interpolation.*
   - (a) Extent of Interpolation.
   - (b) Method of indicating which difference table is to be used.
   - (c) Increase of Respondents from left to right only.
   - (d) Use of Differential instead of Difference.
   - (e) Value chosen for the 'Advance.'
3. *Unit of Angle.*
4. *Tables which at present are rare or wanting.*
5. *List of Mathematical tables referred to and illustrated.*

Before considering these topics seriatim I should like to emphasise the importance to the computer of the arrangement and typography of the mathematical tables which he employs by quoting the authoritative words of Dr Glaisher in his article on Mathematical Tables in the *Encyclopædia Britannica* :—

'The arrangement of a table on the page and all typographical details—such as the shape of the figures, their spacing, the thickness and placing of the rules, the colour and quality of the paper, etc.—are of the highest importance, as the computer has to spend hours with his eyes fixed upon the book; and the efforts of eye and brain required in finding the right numbers amidst a mass of figures on a page and in taking them out accurately, when the computer is tired as well as when he is fresh, are far more trying than the mechanical action of simple reading.'

It might be supposed, the matter being of so much moment, that compilers of tables would long ago have carefully considered it, and that by now a settled procedure would have been adopted. This, however, is very far from being the case: a state of affairs which may perhaps be explained as resulting from the fact that mathematical tables are after all not very numerous, and come from many different sources, alike as regards compilers and printers, thus affording no one person any extended experience in regard to their production.

The occasion of a gathering of computers, therefore, seems to be a good opportunity for bringing forward a number of definite questions in regard to different ways of arranging such tables, with a view to eliciting expressions of opinion concerning their relative merits from those most nearly affected.

I. Typography

(a) *Projecting versus Uniform-sized Figures.*—In regard to this point De Morgan says in his Preface to Barlow's tables :—

'I had long been satisfied that the old numeral symbols, in which most of the figures had heads or tails, were many times more legible than those of uniform height, introduced,

(Moving the decimal point to the RIGHT in n is equivalent to moving it to the LEFT in $1/n$)

Reciprocals

Huntington

10	9	8	7	6	5	4	3	2	1	0	n	Tenths of the Tabular Difference 1	2	3	4	5
.1000	.1010	.1020	.1031	.1042	.1053	.1064	.1075	.1087	.1099	.1111	9.	1	2	3	4	6
.1111	.1124	.1136	.1149	.1163	.1176	.1190	.1205	.1220	.1235	.1250	8.	1	3	4	6	7
.1250	.1266	.1282	.1299	.1316	.1333	.1351	.1370	.1389	.1408	.1429	7.	2	4	5	7	9
.1429	.1449	.1471	.1493	.1515	.1538							2	4	7	9	11
					.1538	.1562	.1587	.1613	.1639	.1667	6.	3	5	8	10	13
.1667	.1669	.1672	.1675	.1678	.1681	.1684	.1686	.1689	.1692	.1695	5.9	0	1	1	1	1
.1695	.1698	.1701	.1704	.1706	.1709	.1712	.1715	.1718	.1721	.1724	5.8	0	1	1	1	1
.1724	.1727	.1730	.1733	.1736	.1739	.1742	.1745	.1748	.1751	.1754	5.7	0	1	1	1	2
.1754	.1757	.1761	.1764	.1767	.1770	.1773	.1776	.1779	.1783	.1786	5.6	0	1	1	1	2
.1786	.1789	.1792	.1795	.1799	.1802	.1805	.1808	.1812	.1815	.1818	5.5	0	1	1	1	2
.1818	.1821	.1825	.1828	.1832	.1835	.1838	.1842	.1845	.1848	.1852	5.4	0	1	1	1	2
.1852	.1855	.1859	.1862	.1866	.1869	.1873	.1876	.1880	.1883	.1887	5.3	0	1	1	1	2
.1887	.1890	.1894	.1898	.1901	.1905	.1908	.1912	.1916	.1919	.1923	5.2	0	1	1	1	2
.1923	.1927	.1931	.1934	.1938	.1942	.1946	.1949	.1953	.1957	.1961	5.1	0	1	1	2	2
.1961	.1965	.1969	.1972	.1976	.1980	.1984	.1988	.1992	.1996	.2000	5.0	0	1	1	2	2
.2000	.2004	.2008	.2012	.2016	.2020	.2024	.2028	.2033	.2037	.2041	4.9	0	1	1	2	2
.2041	.2045	.2049	.2053	.2058	.2062	.2066	.2070	.2075	.2079	.2083	4.8	0	1	1	2	2
.2083	.2088	.2092	.2096	.2101	.2105	.2110	.2114	.2119	.2123	.2128	4.7	0	1	1	2	2
.2128	.2132	.2137	.2141	.2146	.2151	.2155	.2160	.2165	.2169	.2174	4.6	0	1	1	2	2
.2174	.2179	.2183	.2188	.2193	.2198	.2203	.2208	.2212	.2217	.2222	4.5	0	1	1	2	2
.2222	.2227	.2232	.2237	.2242	.2247	.2252	.2257	.2262	.2268	.2273	4.4	1	1	2	2	3
.2273	.2278	.2283	.2288	.2294	.2299	.2304	.2309	.2315	.2320	.2326	4.3	1	1	2	2	3
.2326	.2331	.2336	.2342	.2347	.2353	.2358	.2364	.2370	.2375	.2381	4.2	1	1	2	2	3
.2381	.2387	.2392	.2398	.2404	.2410	.2415	.2421	.2427	.2433	.2439	4.1	1	1	2	2	3
.2439	.2445	.2451	.2457	.2463	.2469	.2475	.2481	.2488	.2494	.2500	4.0	1	1	2	2	3

LOGARITHMS OF NUMBERS.

M'Aulay.

225	0	1	2	3	4	5	6	7	8	9
225	35 218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	003	021	040	059	078	097	116	135	154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	014	033	051	070	088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	005	023	041	058	076
246	39 094	111	129	146	164	182	199	217	235	252
247	·270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777

220″ =0° 3′ 40″
230″ =0° 3′ 50″
240″ =0° 4′ 0″

0°	37′	38′	39′	40′	41′		19	18	17
″	″	″	″	″	″	1	1·9	1·8	1·7
						2	3·8	3·6	3·4
0	2220	2280	2340	2400	2460	3	5·7	5·4	5·1
10	30	90	50	10	70	4	7·6	7·2	6·8
20	40	2300	60	20	80	5	9·5	9·0	8·5
						6	11·4	10·8	10·2
30	2250	2310	2370	2430	2490	7	13·3	12·6	11·9
40	60	20	80	40	2500	8	15·2	14·4	13·6
50	70	30	90	50	10	9	17·1	16·2	15·3

9

FIG. 1.

I believe, by Dr Hutton. From the time when the reprint of Lalande appeared (about twelve months ago) I have heard no one contest this position; and the present work will show that it is true of a *heavy* page (as the printers call it) as of the one in which there are fewer figures.'

Despite this opinion, written in 1839, both styles of figures are still to be found, and no general agreement seems to have been reached as to which is the better kind. The various examples in different tables which the author has been able to examine certainly suggest the correctness of De Morgan's view. In Fig. 1 specimens of the two different styles of type, both taken from well-known tables, will be found side by side.

(b) *Large or Small Type.*—As economy of space is always a consideration, it becomes a question whether the available space is best used by having the type so large as almost completely to fill it, or whether it is better to use smaller type and thus find room for considerable spaces between the entries. Two very good examples of the opposite extremes will be found in Fig. 1. It must always be remembered that in the case of tables of figures, where each symbol has to be visualised separately, the type must be more legible than in the case of reading matter, where only collections of symbols, *i.e.* words, have to be visualised. Dr Sang, the famous computer of logarithms held a decided opinion in regard to this point, and says in the Preface to his books of tables :—

'Many trials and long experience have shown that the legibility of a mass of figures depends much more upon the separation of the figures from each other than upon their shapes; and the eminent firm of Millar and Richard, type-founders, have supplied a fount of types in which the figures are kept much more apart than is usual. The result has been a page of tabular matter which does not strain the eye, and which will compare favourably in point of legibility with any other type.' (See Fig. 2.)

Becker and Van Orstrand.

u	sin u	∞ F₀′	cos u	∞ F₀′	log sin u	∞ F₀′	log cos u	∞ F₀′	u
									° ′ ″
0.0850	0.08490	10,0	0.99639	0,8	8.92890	51,0	9.99843	0,4	4 52 12.51
.0851	.08500		.99638	0,8	.92941	50,9	.99843		4 52 33.14
.0852	.08510		.99637	0,9	.92991	50,9	.99842		4 52 53.76
.0853	.08520		.99636		.93042	50,8	.99842		4 53 14.39
.0854	.08530		.99636		.93093	50,7	.99841		4 53 35.01
0.0855	0.08540	10,0	0.99635	0,9	8.93144	50,7	9.99841	0,4	4 53 55.64
.0856	.08550		.99634		.93194	50,6	.99841		4 54 16.27
.0857	.08560		.99633		.93245	50,6	.99840		4 54 36.89
.0858	.08569		.99632		.93295	50,5	.99840		4 54 57.52
.0859	.08579		.99631		.93346	50,4	.99840		4 55 18.15
0.0860	0.08589	10,0	0.99630	0,9	8.93396	50,4	9.99839	0,4	4 55 38.77
.0861	.08599		.99630		.93447	50,3	.99839		4 55 59.40
.0862	.08609		.99629		.93497	50,3	.99838		4 56 20.03
.0863	.08619		.99628		.93547	50,2	.99838		4 56 40.65
.0864	.08629		.99627		.93597	50,1	.99838		4 57 01.28
0.0865	0.08639	10,0	0.99626	0,9	8.93647	50,1	9.99837	0,4	4 57 21.91
.0866	.08649		.99625		.93697	50,0	.99837		4 57 42.53
.0867	.08659		.99624		.93747	50,0	.99837		4 58 03.16
.0868	.08669		.99624		.93797	49,9	.99836		4 58 23.79
.0869	.08679		.99623		.93847	49,9	.99836		4 58 44.41

Sang.

	0	1	2	3	4	5	6	7	8	9
9080	0858	0906	0954	1002	1050	1098	1145	1193	1241	1289
81	1337	1385	1432	1480	1528	1576	1624	1672	1719	1767
82	1815	1863	1911	1958	2006	2054	2102	2150	2198	2245
83	2293	2341	2389	2437	2484	2532	2580	2628	2676	2723
84	2771	2819	2867	2915	2962	3010	3058	3106	3154	3202
85	3249	3297	3345	3393	3441	3488	3536	3584	3632	3680
86	3727	3775	3823	3871	3919	3966	4014	4062	4110	4157
87	4205	4253	4301	4349	4396	4444	4492	4540	4588	4635
88	4683	4731	4779	4827	4874	4922	4970	5018	5065	5113
89	5161	5209	5257	5304	5352	5400	5448	5495	5543	5591
9090	5639	5687	5734	5782	5830	5878	5925	5973	6021	6069
91	6117	6164	6212	6260	6308	6355	6403	6451	6499	6547
92	6594	6642	6690	6738	6785	6833	6881	6929	6976	7024
93	7072	7120	7167	7215	7263	7311	7358	7406	7454	7502
94	7549	7597	7645	7693	7741	7788	7836	7884	7932	7979
95	8027	8075	8123	8170	8218	8266	8314	8361	8409	8457
96	8505	8552	8600	8648	8695	8743	8791	8839	8886	8934
97	8982	9030	9077	9125	9173	9221	9268	9316	9364	9412
98	9459	9507	9555	9603	9650	9698	9746	9793	9841	9889
99	9937	9984	♦032	♦080	♦128	♦175	♦223	♦271	♦318	♦366
,959	0	1	2	3	4	5	6	7	8	9

1	47
2	94
3	141
4	188
5	235
6	282
7	329
8	376
9	423

147

Fig. 2.

(c) *Grouping of Columns.*—The various columns headed by the numbers from 1-9, which are necessary in many tabulations, are usually divided for greater legibility either into two or into three groups, separated by heavy lines or wide spaces. For an example of the former see 'Chambers,' Fig. 3, for an example of the latter see 'M'Aulay,' Fig. 1. The author's own experience is strongly in favour of a division into three groups, because in the case of this arrangement, it soon becomes unnecessary to look at the numbers at the top of the columns. In the case where the division is into two groups, the eye cannot be safely trusted to identify a column even after considerable practice.

The question whether the separation of the columns is best effected by lines or by blank spaces is one of some importance. Dr Sang, in the Preface already alluded to, says :—

'While editing Shortrede's Logarithmic Tables, I received a page without the usual rules because the printer's stock of them had been exhausted. The superiority of the white spaces was so manifest that I at once adopted them in the subsequent table of Antilogarithms printed in that work. The same plan is followed here, and the only line used is a thin one to separate the difference tables from the body of the page.' (See Fig. 2 and for an example of the opposite kind see Fig. 5.)

(d) *The Method of indicating a Change in a Leading Figure.*—The methods employed to indicate a change in a leading figure are very various. Sometimes a single asterisk or corresponding mark is used only at the first group of figures where the change occurs; in other cases each one of the groups of figures affected is marked with an asterisk. An example of the former kind will be found in 'Bauschinger and Peters,' Fig. 4, and as in this case the eye can only reach the wrong set of leading figures by passing the asterisk on its way, it may be that a single asterisk is sufficient in all

NAPIERIAN LOGARITHMS

of Numbers from 0·1 to 5·09; with Subsidiary Table.

Knott.

Number.	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
0·1	$\bar{3}$·6974	7927	8797	9598	$\bar{0}$339	$\bar{1}$029	$\bar{1}$674	$\bar{2}$280	$\bar{2}$852	$\bar{3}$393									
0·2	$\bar{2}$·3906	4393	4859	5303	5729	6137	6529	6907	7270	7621									
0·3	·7960	8238	8606	8913	9212	9502	9783	$\bar{0}$057	$\bar{0}$324	$\bar{0}$584									
0·4	$\bar{1}$·0837	1084	1325	1560	1790	2015	2235	2450	2660	2866									
0·5	3068	3267	3461	3651	3838	4022	4202	4379	4553	4724									
0·6	4892	5057	5220	5380	5537	5692	5845	5995	6143	6289									
0·7	6433	6575	6715	6853	6989	7123	7256	7386	7515	7643									
0·8	7769	7893	8015	8137	8256	8375	8492	8607	8722	8835									
0·9	8946	9057	9166	9274	9381	9487	9592	9695	9798	9899									
1·0	0·0000	0100	0198	0296	0392	0488	0583	0677	0770	0862									
1·1	0953	1044	1133	1222	1310	1398	1484	1570	1655	1740	9	17	26	35	44	52	61	70	78
1·2	1823	1906	1989	2070	2151	2231	2311	2390	2469	2546	8	16	24	32	40	48	56	64	72
1·3	2624	2700	2776	2852	2927	3001	3075	3148	3221	3293	7	15	22	30	37	45	52	59	67
1·4	3365	3436	3507	3577	3646	3716	3784	3853	3920	3988	7	14	21	28	35	41	48	55	62
1·5	4055	4121	4187	4253	4318	4383	4447	4511	4574	4637	6	13	19	26	32	39	45	52	58
1·6	4700	4762	4824	4886	4947	5008	5068	5128	5188	5247	6	12	18	24	30	36	42	48	55
1·7	5306	5365	5423	5481	5539	5596	5653	5710	5766	5822	6	11	17	24	29	34	40	46	52
1·8	5878	5933	5988	6043	6098	6152	6206	6259	6313	6366	5	11	16	22	27	32	38	43	49
1·9	6419	6471	6523	6575	6627	6678	6729	6780	6831	6881	5	10	15	20	26	31	36	41	46
2·0	6931	6981	7031	7080	7129	7178	7227	7275	7324	7372	5	10	15	20	24	29	34	39	44

Subsidiary Table.

No.	Nap. Log.	No.	Nap. Log.
6	1·7918	40	3·6889
7	1·9459	50	3·9120
8	2·0794	60	4·0943
9	2·1972	70	4·2485
10	2·3026	80	4·3820
20	2·9957	90	4·4998
30	3·4012	100	4·6052

3250

Chambers.

No.		0	1	2	3	4	5	6	7	8	9	Diff.
3250	511	8834	8967	9101	9234	9368	9502	9635	9769	9903	$\overline{0036}$	
51	512	0170	0303	0437	0570	0704	0838	0971	1105	1238	1372	
52		1505	1639	1772	1906	2040	2173	2307	2440	2574	2707	
53		2841	2974	3108	3241	3375	3508	3642	3775	3909	4042	
54		4175	4309	4442	4576	4709	4843	4976	5110	5243	5377	
55		5510	5643	5777	5910	6044	6177	6310	6444	6577	6711	
56		6844	6977	7111	7244	7377	7511	7644	7778	7911	8044	
57		8178	8311	8444	8578	8711	8844	8978	9111	9244	9377	
58		9511	9644	9777	9911	$\overline{0044}$	$\overline{0177}$	$\overline{0311}$	$\overline{0444}$	$\overline{0577}$	$\overline{0710}$	
59	513	0844	0977	1110	1243	1377	1510	1643	1776	1910	2043	
60		2176	2309	2442	2576	2709	2842	2975	3108	3242	3375	
3261		3508	3641	3774	3908	4041	4174	4307	4440	4573	4706	133
62		4840	4973	5106	5239	5372	5505	5638	5771	5905	6038	1 13
63		6171	6304	6437	6570	6703	6836	6969	7102	7235	7368	2 27
64		7502	7635	7768	7901	8034	8167	8300	8433	8566	8699	3 40
65		8832	8965	9098	9231	9364	9497	9630	9763	9896	$\overline{0029}$	4 53
66	514	0162	0295	0428	0561	0694	0827	0960	1093	1225	1358	5 67
67		1491	1624	1757	1890	2023	2156	2289	2422	2555	2688	6 80
68		2820	2953	3086	3219	3352	3485	3618	3751	3883	4016	7 93
69		4149	4282	4415	4548	4681	4813	4946	5079	5212	5345	8 106
70		5478	5610	5743	5876	6009	6142	6274	6407	6540	6673	9 120
3271		6805	6938	7071	7204	7336	7469	7602	7735	7867	8000	
72		8133	8266	8398	8531	8664	8797	8929	9062	9195	9327	
73		9460	9593	9725	9858	9991	$\overline{0123}$	$\overline{0256}$	$\overline{0389}$	$\overline{0521}$	$\overline{0654}$	
74	515	0787	0919	1052	1185	1317	1450	1583	1715	1848	1980	
75		2113	2246	2378	2511	2643	2776	2909	3041	3174	3306	

FIG. 3.

cases to prevent mistakes, although this may be doubted. An example of the latter arrangement—in which each group is marked—from 'Erskine Scott' will be found in Fig. 5 for comparison. Another method of indicating the change is to draw a bar either above or below the entries affected. Two examples are given, in one of which the bar is drawn over all the figures in each entry (see 'Chambers,' Fig. 3); while in the other a short bar is drawn over the first two figures only, a neater if less arresting method (see 'Knott' in the same figure). Probably the best way of indicating the change is by the use of heavier type, a good example of which will be found in 'M'Aulay' (Fig. 1), but some may prefer the nokta introduced by Sang (see Fig. 2), when he edited Shortrede's Tables. Sang states in the Preface to his own tables that :—

'M. V. Bagay, in his *Tables Hydrographiques*, Paris, 1829, adopted a distinct mark ⊙ to indicate a change in the leading figures. This idea was followed up in Shortrede's Tables, an enlarged Arab nokta ◆ being substituted for the European nulla, and repeated along the line. This method of indicating the change is followed in the present work. Farther, at the suggestion of Mr William Thomas Thomson, manager of the Standard Life Assurance Company, the first three figures of the logarithm are placed below the last line of the page whenever a change occurs there.'

The addition of the leading figures at the bottom of a page on which they would otherwise not appear (see Fig. 2), is certainly a practice to be commended.

In the usual arrangement of the asterisk it is customary to print it as an addition to the group of figures, but Sang makes his nokta a *substitute* for the zero digit, and thus no additional space is required, which in many cases constitutes an important advantage.

(e) *The use of Coloured Paper or Ink.*—Coloured or tinted

	18° 57′				18° 58′			
d.	613	685	685	72	613	685	685	72
″	sin	tang	cotg	cos	sin	tang	cotg	cos
	9.511	9.535	0.464	9.975	9.511	9.536	0.463	9.975
0	53968	73930	26070	80038	90744	15046	84954	75698
1	54581	74615	25385	79966	91357	15731	84269	75626
2	55194	75301	24699	79893	91969	16416	83584	75553
3	55807	75986	24014	79821	92582	17101	82899	75481
4	56421	76672	23328	79749	93194	17786	82214	75409
5	57034	77357	22643	79676	93807	18471	81529	75336
6	57647	78043	21957	79604	94420	19156	80844	75264
7	58260	78728	21272	79532	95032	19841	80159	75192
8	58873	79414	20586	79459	95645	20525	79475	75119
9	59486	80099	19901	79387	96257	21210	78790	75047
10	60100	80785	19215	79315	96870	21895	78105	74975
11	60713	81470	18530	79243	97482	22580	77420	74902
12	61326	82156	17844	79170	98095	23265	76735	74830
13	61939	82841	17159	79098	98707	23950	76050	74757
14	62552	83526	16474	79026	99320	24635	75365	74685
15	63165	84212	15788	78953	99932	25320	74680	74613
16	63778	84897	15103	78881	*00545	26005	73995	74540
17	64391	85583	14417	78809	01157	26690	73310	74468
18	65004	86268	13732	78736	01770	27374	72626	74395
19	65617	86953	13047	78664	02382	28059	71941	74323
20	66230	87639	12361	78592	02995	28744	71256	74251
21	66843	88324	11676	78519	03607	29429	70571	74178
22	67456	89009	10991	78447	04220	30114	69886	74106
23	68069	89695	10305	78375	04832	30799	69201	74034
24	68682	90380	09620	78302	05445	31483	68517	73961
25	69295	91065	08935	78230	06057	32168	67832	73889
26	69908	91751	08249	78158	06669	32853	67147	73816
27	70521	92436	07564	78085	07282	33538	66462	73744
28	71134	93121	06879	78013	07894	34223	65777	73672
29	71747	93806	06194	77941	08507	34907	65093	73599
30	72360	94492	05508	77869	09119	35592	64408	73527
31	72973	95177	04823	77796	09731	36277	63723	73454
32	73586	95862	04138	77724	10344	36962	63038	73382
33	74199	96547	03453	77652	10956	37646	62354	73310
34	74812	97233	02767	77579	11568	38331	61669	73237
35	75425	97918	02082	77507	12181	39016	60984	73165
36	76038	98603	01397	77435	12793	39700	60300	73092
37	76651	99288	00712	77362	13405	40385	59615	73020
38	77263	99974	00026	77290	14017	41070	58930	72948
39	77876	*00659	*99341	77218	14630	41754	58246	72875
40	78489	01344	98656	77145	15242	42439	57561	72803
41	79102	02029	97971	77073	15854	43124	56876	72730
42	79715	02714	97286	77001	16466	43808	56192	72658
43	80328	03399	96601	76928	17079	44493	55507	72586
44	80940	04085	95915	76856	17691	45178	54822	72513
45	81553	04770	95230	76783	18303	45862	54138	72441
46	82166	05455	94545	76711	18915	46547	53453	72368

Bauschinger and Peters.

FIG. 4.

paper is sometimes used in books of mathematical tables for the purpose of distinction; *e.g.* in Erskine Scott's tables of logarithms and antilogarithms, the logarithms are printed on white pages, the antilogarithms on green.

Occasionally, also, a coloured paper is employed as being less trying for the eye. Although in this way a softer effect is no doubt obtained, and some experienced computers assert that they prefer coloured or at least tinted paper, it may be questioned whether the advantage gained is not more than outweighed by the ensuing reduction of legibility. For an example of printing on a green paper the reader may turn to Fig. 5, where part of a page of Erskine Scott's antilogarithm table is reproduced. The best plan, of course, as has been pointed out, would be to have white printing on black paper. By this means only the objects which the eye requires to sec, namely the figures, are illuminated. The background is left dark, so that the total amount of light entering the eye, and the consequent fatigue, is much diminished. Although for technical reasons it is impossible to print in this manner, it is an interesting fact that recently makers of counting machines have begun to make the change, and the author can bear personal testimony in the case of one apparatus at least to the decided advantage that has resulted.

As regards the use of coloured inks the British Association Report on the 'Influence of School Books upon Eyesight' (1913) asserts that: 'The use of coloured inks for reading matter is strongly to be deprecated, especially the use of more than one colour on a page.' An example of a single colour (red) is afforded by the reproduction of part of a page of 'Douglas' in Fig. 6; and of two colours (red and black) by the reproduction of part of another page in the same figure. The singularly confusing effect of the two colours will probably be readily acknowledged.

There is no objection, of course, to the mere drawing of a

Erskine Scott.

Log	0		1	2	3	4	5	6	7	8	9
·3550	226	46	47	47	48	49	49	50	50	51	51
·3551	226	52	52	53	53	54	54	55	55	56	56
·3552	226	57	57	58	58	59	59	60	61	61	62
·3553	226	62	63	63	64	64	65	65	66	66	67
·3554	226	67	68	68	69	69	70	70	71	71	72
·3555	226	73	73	74	74	75	75	76	76	77	77
·3556	226	78	78	79	79	80	80	81	81	82	82
·3557	226	83	84	84	85	85	86	86	87	87	88
·3558	226	88	89	89	90	90	91	91	92	92	93
·3559	226	93	94	94	95	96	96	97	97	98	98
·3560	226	99	99	*00	*00	*01	*01	*02	*02	*03	*03
·3561	227	04	04	05	05	06	06	07	08	08	09
·3562	227	09	10	10	11	11	12	12	13	13	14
·3563	227	14	15	15	16	16	17	17	18	19	19
·3564	227	20	20	21	21	22	22	23	23	24	24
·3565	227	25	25	26	26	27	27	28	28	29	30
·3566	227	30	31	31	32	32	33	33	34	34	35
·3567	227	35	36	36	37	37	38	38	39	39	40
·3568	227	41	41	42	42	43	43	44	44	45	45
·3569	227	46	46	47	47	48	48	49	49	50	50
·3570	227	51	52	52	53	53	54	54	55	55	56
·3571	227	56	57	57	58	58	59	59	60	60	61
·3572	227	61	62	63	63	64	64	65	65	66	66
·3573	227	67	67	68	68	69	69	70	70	71	71
·3574	227	72	72	73	74	74	75	75	76	76	77
·3575	227	77	78	78	79	79	80	80	81	81	82
·3576	227	82	83	83	84	85	85	86	86	87	87
·3577	227	88	88	89	89	90	90	91	91	92	92
·3578	227	93	93	94	95	95	96	96	97	97	98
·3579	227	98	99	99	*00	*00	*01	*01	*02	*02	*03
·3580	228	03	04	04	05	06	06	07	07	08	08
·3581	228	09	09	10	10	11	11	12	12	13	13
·3582	228	14	14	15	16	16	17	17	18	18	19
·3583	228	19	20	20	21	21	22	22	23	23	24
·3584	228	24	25	25	26	27	27	28	28	29	29
·3585	228	30	30	31	31	32	32	33	33	34	34
·3586	228	35	35	36	37	37	38	38	39	39	40
·3587	228	40	41	41	42	42	43	43	44	44	45
·3588	228	45	46	47	47	48	48	49	49	50	50
·3589	228	51	51	52	52	53	53	54	54	55	55

Fig. 5.

red line or the like, as a special reminder at some part of a page, and this is occasionally done with an excellent result.

(f) *Provision of a Lateral Index.*—Very few tables are published in which a lateral index is provided, probably on account of the additional expense; but this is to be regretted, as in most cases such an index affords a great saving of time. For example, in the case of a 5-figure logarithm table, where at each opening of the book one thousand values of the argument are tabulated, such an index enables the computer to turn to the page required immediately. In Holman's Collection of Mathematical Tables a lateral index is not provided by the publisher, but the making of one by the purchaser for his own use is facilitated by the following device. One of the pages is set apart for the purpose, and has printed upon it the leading figures which are to appear at the side of the book; and this page when cut up furnishes the necessary tabs for the index. Such a plan, which might well be imitated in other cases, does not increase the cost of production, and the slight amount of trouble entailed in the construction of the index by the computer is far more than repaid him in the consequent saving of time.

(g) *Paper, Printing and Binding.*—In regard to printing all that is comprehended under the term 'workmanship,' that is, the alignment of the type, the uniformity of the impression, etc., must of course be faultless if the book is to be considered satisfactory. With possibly one or two exceptions, consisting only of small and cheap publications, the workmanship of books of mathematical tables appears to be of a high standard. The same, however, cannot always be said of their binding. Too often this is so stiff and poor as to prevent the pages from being turned over readily and from lying flat when the books are opened. Such a defect is a very serious one in the case of a book of tables, the pages of which are, of necessity, being constantly turned

LOGARITHMS OF SINES.

	0′	10′	20′	30′	40′	50′	60′		Differences. 1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	– ∞	$\bar{3}$·4637	$\bar{3}$·7648	$\bar{3}$·9408	$\bar{2}$·0658	$\bar{2}$·1627	$\bar{2}$·2419	89°	For small angles of n minutes of arc, log sine n' or log cosine $(90° - n')$ $= \log n + \bar{4}·4637$ Differences vary so rapidly here that tabulation is impossible.								
1	$\bar{2}$·2419	$\bar{2}$·3088	$\bar{2}$·3668	$\bar{2}$·4179	$\bar{2}$·4637	$\bar{2}$·5050	$\bar{2}$·5428	88									
2	$\bar{2}$·5428	$\bar{2}$·5776	$\bar{2}$·6097	$\bar{2}$·6397	$\bar{2}$·6677	$\bar{2}$·6940	$\bar{2}$·7188	87									
3	$\bar{2}$·7188	$\bar{2}$·7423	$\bar{2}$·7645	$\bar{2}$·7857	$\bar{2}$·8059	$\bar{2}$·8251	$\bar{2}$·8436	86									
4	$\bar{2}$·8436	$\bar{2}$·8613	$\bar{2}$·8783	$\bar{2}$·8946	$\bar{2}$·9104	$\bar{2}$·9256	$\bar{2}$·9403	85									
5	$\bar{2}$·9403	$\bar{2}$·9545	$\bar{2}$·9682	$\bar{2}$·9816	$\bar{2}$·9945	$\bar{1}$·0070	$\bar{1}$·0192	84									
6	$\bar{1}$·0192	$\bar{1}$·0311	$\bar{1}$·0426	$\bar{1}$·0539	$\bar{1}$·0648	$\bar{1}$·0755	$\bar{1}$·0859	83									
7	$\bar{1}$·0859	$\bar{1}$·0961	$\bar{1}$·1060	$\bar{1}$·1157	$\bar{1}$·1252	$\bar{1}$·1345	$\bar{1}$·1436	82	10	19	29	39	48	58	67	77	87
8	$\bar{1}$·1436	$\bar{1}$·1525	$\bar{1}$·1612	$\bar{1}$·1697	$\bar{1}$·1781	$\bar{1}$·1863	$\bar{1}$·1943	81	8	17	25	34	42	51	59	68	76
9	$\bar{1}$·1943	$\bar{1}$·2022	$\bar{1}$·2100	$\bar{1}$·2176	$\bar{1}$·2251	$\bar{1}$·2324	$\bar{1}$·2397	80	8	15	23	30	38	45	53	61	68
10°	$\bar{1}$·2397	$\bar{1}$·2468	$\bar{1}$·2538	$\bar{1}$·2606	$\bar{1}$·2674	$\bar{1}$·2740	$\bar{1}$·2806	79	7	14	20	27	34	41	48	55	62
11	·2806	·2870	·2934	·2997	·3058	·3119	·3179	78	6	12	19	25	31	37	44	50	56
12	·3179	·3238	·3296	·3353	·3410	·3466	·3521	77	6	11	17	23	29	34	40	46	51
13	·3521	·3575	·3629	·3682	·3734	·3786	·3837	76	5	11	16	21	26	32	37	42	47
14	·3837	·3887	·3937	·3986	·4035	·4083	·4130	75	5	10	15	20	24	29	34	39	44
15	$\bar{1}$·4130	$\bar{1}$·4177	$\bar{1}$·4223	$\bar{1}$·4269	$\bar{1}$·4314	$\bar{1}$·4359	$\bar{1}$·4403	74	5	9	14	18	23	27	32	36	41
16	·4403	·4447	·4491	·4533	·4576	·4618	·4659	73	4	9	13	17	21	26	30	34	38
17	·4659	·4700	·4741	·4781	·4821	·4861	·4900	72	4	8	12	16	20	24	28	32	36
18	·4900	·4939	·4977	·5015	·5052	·5090	·5126	71	4	8	11	15	19	23	26	30	34
19	·5126	·5163	·5199	5235	·5270	5306	5341	70°	4	7	11	14	18	21	25	29	32

ANTI-LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3

Fig. 6.

over by the computer; while a page which does not lie flat presents different parts of its surface at different angles to the eye, thus throwing an unnecessary strain on the latter.

The constant turning of the pages, to which reference has just been made, has, of course, a bad effect on the paper, and unless this be of a sort specially calculated to resist wear and tear deterioration soon begins. It is unfortunately the case that the paper in too many modern books of tables is such as should never have been used for the purpose, and publishers should see to it that the paper they employ is of exceptionally good quality in view of the necessarily severe treatment to which it will be subjected. The attention of the author was recently called to a case, in which the owner of an old book of mathematical tables had wisely preferred to have it rebound, rather than purchase a new copy of the contemporaneous edition, because of the marked inferiority of the paper of the latter.

II. Interpolation

(a) *Extent of Interpolation.*—By this heading is meant the extent of the interpolation required from the computer in order to obtain the full accuracy which the table is capable of giving. The two extremes are to be found, on the one hand in those tables where no interpolation at all is required, and on the other in those tables in which interpolation for two of the figures of the argument is needed; or it may be in which linear interpolation is insufficient, and second differences have to be taken into account. Examples of the former are to be found in Erskine Scott's two tables of 4-figure and 5-figure logarithms and antilogarithms, in which the arguments are given to 4 and to 5 figures respectively (see Fig. 5). The effect of this method of tabulation is to produce a

I.—LOGARITHMS OF NUMBERS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											Mean Differences.								
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743	42	83	125	166	208	248	290	331	373
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555	38	76	114	152	190	226	264	302	340
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059	35	70	105	140	175	208	243	278	313
13	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301	32	65	97	129	162	193	225	257	290
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319	30	60	90	120	150	179	209	239	269
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140	28	56	84	112	140	168	196	224	252
16	20412	20683	20952	21219	21484	21748	22011	22272	22531	22789	26	53	79	105	132	158	184	210	237
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285	25	50	74	99	124	149	174	199	223
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646	23	47	70	94	117	141	164	188	211
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885	22	45	67	89	111	134	156	178	201
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015	21	42	64	85	106	127	148	170	191
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044	20	40	61	81	101	121	141	162	182
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984	19	39	58	77	97	116	135	154	174
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840	19	37	56	74	93	111	130	148	167
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620	18	35	53	71	89	106	124	142	159
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330	17	34	51	68	85	102	119	136	153
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975	16	33	49	66	82	98	115	131	148
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560	16	32	47	63	79	95	111	126	142
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090	15	30	46	61	76	91	106	122	137
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567	15	29	44	59	74	88	103	118	132
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996	14	29	43	57	72	86	100	114	129
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379	14	28	41	55	69	83	97	110	125
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720	13	27	40	54	67	80	94	107	121
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020	13	26	39	52	65	78	91	104	117
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283	13	25	38	50	63	76	88	101	113
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509	12	24	37	49	61	73	85	98	110
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703	12	24	36	48	60	71	83	95	107
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864	12	23	35	46	58	70	81	93	104
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995	11	23	34	45	57	68	79	90	102
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097	11	22	33	44	55	66	77	88	99

Fig. 7.

Note.—The doubling of the lines enclosing the columns under '5' is not in the original, but was done by the writer in his private copy, from which the above was reproduced.

serious increase in the size of the book, and to necessitate much turning over of pages; but a table of this kind is very useful, when, as is often the case, a constant value has to be added to each of the respondents, or when these have to be multiplied or divided by some simple integer. In such cases the absence of interpolation renders it easy to perform the necessary arithmetical processes mentally without fear of mistake, while the numbers are being picked out. An example of the opposite extreme, in which interpolation for two of the figures of the argument is necessary, is to be found in Dale's Tables (see Fig. 7). The resulting compactness is very marked, but it may well be doubted if such compactness is not too dearly paid for by the increased labour it entails upon the computer.

The method of tabulating the tenths of the differences from 1-5 only, and of obtaining the others by subtraction from the succeeding respondent, which is to be found, for instance, in Huntington's tables (see Fig. 1), is a plan which not only saves space, but which also reduces the magnitude of the largest of the proportional parts by half. This means that in many cases the values of the proportional parts never rise above a single figure, which simplifies the work of interpolation and considerably reduces liability to error. Against this one has to put the fact that through force of habit addition may sometimes be used unwittingly instead of subtraction; but in the author's experience there is no danger of this after a little practice.

(b) *Method of indicating which Difference Table is to be used.*—In many cases a table of proportional parts is provided at the end of each row, to every respondent of which it will apply very approximately. In order to increase the accuracy the row is sometimes broken across, so that different tables of proportional parts may be used for its two portions (see 'Huntington,' Fig. 1). This idea is carried still further in the tables compiled by Woodward, who, by means of a

N.	L.	0	1	2	3	4	5	6	7	8	9	
240	38	021	039	057	075	093̄	112	130	148	166	184	*b*
1		202	220	238	256	274	292	310	328	346	364	
2		382̲	399	417	435	453	471	489	507	525	543	
3		561̲	578	596	614	632	650	668	686̲	703	721	
4		739	757	775̲	792	810	828	846̲	863	881	899	
5	38	917̲	934	952	970̲	987	·005	·023	·041̲	·058	·076	
6	39	094̲	111	129̲	146	164	182̲	199	217	235̲	252	
7		270̲	287	305̲	322	340	358̲	375	393̲	410	428̲	*b*
8		445̄	463	480̄	498	515̄	533	550̄	568	585	602̄	*c*
9		620	637̄	655	672̄	690	707	724̄	742	759̄	777	
250	39	794	811̄	829	846	863̄	881	898	915̄	933	950	
1		967̄	985	·002	·019̄	·037	·054	·071	·088̄	·106	·123	
2	40	140	157̄	175	192	209	226	243̄	261	278	295	
3		312	329	346̄	364	381	398	415	432	449	466	
4		483	500̄	518	535	552	569	586	603	620	637	
5	40	654	671	688	705	722	739	756	773	790	807	
6		824	841	858	875	892	909	926	943	960̲	976	
7		993	·010	·027	·044	·061	·078	·095̲	·111	·128	·145	
8	41	162	179	196̲	212	229	246	263	280̲	296	313	
9		330	347̲	363	380	397	414̲	430	447	464	481̲	
260	41	497	514	531̲	547	564	581̲	597	614	631̲	647	
1		664	681̲	697	714	731̲	747	764̲	780	797	814̲	
2		830	847̲	863	880̲	896	913̲	929	946	963̲	979	*c*
3		996	·012̄	·029	·045̄	·062	·078̄	·095	·111	·127̄	·144	*d*
4	42	160̄	177	193̄	210	226̄	243	259	275̄	292	308̄	
5	42	325	341	357̄	374	390	406̄	423	439	455̄	472	
6		488	504̄	521	537	553̄	570	586	602̄	619	635	
7		651	667̄	684	700	716	732̄	749	765	781	797	
8		813̄	830	846	862	878	894̄	911	927	943	959	
9		975	991̄	·008	·024	·040	·056	·072	·088	·104	·120	*d*
N.	L.	0	1	2	3	4	5	6	7	8	9	
		a	2	4	6	8	9	11	13	15	17	
		b	2	4	5	7	9	11	13	14	16	
		c	2	3	5	7	8	10	12	14	15	
		d	2	3	5	6	8	10	11	13	14	

210
·322

270
·431

330
·518

390
·591

450
·653

510
·707

570
·755

630
·799

690
·838

750
·875

810
·908

870
·939

930
·968

FIG. 8.

This page shows the value of the lateral index (see p. 304). Only figures 210 and ·322 are printed on this page, and refer to pages 12 and 13. The others, although the reproduction fails to show it, belong to the succeeding pages.

simple system of letters and lines, refers one to the proper table printed at the foot of the page (see Fig. 8). The author has made some use of Woodward's tables, and has certainly found their ingenious system to be a saving of time. In the ordinary case where the difference tables are placed by themselves, with no indication as to which one is to be employed, the user has to ascertain this by mentally subtracting two respondents, but as only the last figures of these need be subtracted, after all the labour is not great.

(c) *Increase of Respondents from left to right only.*—A small point, but one to which it may be worth while to draw attention, is the arrangement of a table so that the value of the respondents invariably increases from left to right (see 'Huntington,' Fig. 1). It will be found in the case of most tables, that such functions, for example, as cosines and reciprocals have the respondents increasing in the opposite direction, a fact which occasionally gives rise to mistakes in interpolation.

(d) *Use of Differential instead of Difference.*—Becker and Van Orstrand in the Smithsonian Table of Hyberbolic Functions give instead of the difference of successive respondents the value of the differential coefficient of the function (see Fig. 2).

The method and its advantages will be sufficiently indicated by the following extract from their Preface:—

'In the tables which follow, the first derivatives multiplied by ω are tabulated in units of the last decimal place of the tabulated function (except Table VII), and the remaining quantities required in the computation can be found by mere inspection. The higher order of differences will be needed only for a very few arguments at the beginning or end of those tabular values whose numerical magnitudes approach 0 or ∞. For the remaining arguments it will be found that the $\frac{1}{48}$ part of the second difference of $\omega F_n'$ is not great enough to influence the result, and it is therefore sufficient to use

$$F_n = F_0 + n\omega(F_0' + \frac{n}{2}a_0)$$
$$F_{-n} = F_0 - n\omega(F_0' - \frac{n}{2}a_0) \quad . \quad . \quad . \quad . \quad (b),$$

ωa_0 being the mean first difference of $\omega F'$ corresponding to F_0. This formula is rigorous when the third differences are zero. In most cases $\frac{n\omega a_0}{2}$ can be found mentally, and since $\omega(F_0' + \frac{n}{2}a_0)$ is here to be regarded as an interpolated value of $\omega F_0'$, no confusion can arise as to the sign of the correction. It thus becomes almost as easy to include ωa_0 in the computation as to omit it. A convenient rule is: Find by linear interpolation the value $\omega F'$ for one-half the interval $\left(\frac{n}{2}\right)$: multiply this interpolated value by the entire interval (n) and apply the product to the tabular value of the function, either positively or negatively, according as the function is increasing or decreasing. . . .

'The derivative formula (b) with two terms has the advantage of being much more convenient than the difference formula, while the accuracy of the two is the same (five-eighths of a unit) when the derivatives are tabulated to the same order of decimal as the function. In the case of linear interpolation, however, it is in general more accurate to use the differences, the maximum error of the difference formula being one-half of a unit and that of the derivative formula three-fourths of a unit in the next succeeding decimal place. The accuracy of the two formulas is the same when the next succeeding decimal of the derivative is tabulated. The error of the derivative formula is then simply the error of the tabular value, while the error of the difference formula may be =, > or < than that of the tabular value, but is never greater than one-half of a unit.'

(e) *Value chosen for the 'Advance.'*—Evidently the larger the advance the fewer the entries in a given range of the function, and hence the more compact the table. The possible largeness of the advance is limited, however, by the need for so restricting it that the resulting first differences of the function shall always be kept small. It usually happens that this condition is more easily satisfied at one end of the table than at the other, and therefore it may be desirable to alter the value of the advance at some point for this reason. By increasing the value of the advance for the larger values of the argument, Huntington in some of his tables has been able to condense into one page matter which tabulated in the ordinary way would occupy two (see Fig. 1). For a special discussion of this point, see Professor Steggall's paper printed below (p. 319).

Tables of the logarithms of numbers and other like tables are frequently provided with an auxiliary table in which the advance is only one-tenth of that of the main table, for the purpose of supplementing the earlier parts of the latter, where the first differences are apt to be undesirably great. A somewhat different arrangement of tabulation which, however, virtually amounts to this, was carried out by Sang in his logarithm table, and that on a heroic scale. As he explains, speaking of the then existing tables :—

'At the beginning of the seven-place table, the differences are large and numerous, the side tables of proportional parts are crammed into the page, and hence the interpolations can be performed mentally only by experienced computers. As we advance in the table, the differences become smaller and fewer, so that the interpolation is easily managed : the object of the present table is to secure this facility all along. According to Napier's idea, a Logarithmic table, in order to be complete, must extend from some number to its decuple :

from 10,000 to 100,000 as in the usual tables; from 20,000 to 200,000 as in this new table.

'By placing the beginning at 20,000 instead of at 10,000 the differences are halved in magnitude, while the number of them in a page is quartered, so that the labour of the interpolation is very much lessened, and great additional facility is afforded to the computer.'

III. Unit of Angle

Despite the length of time that has elapsed since tables of trigonometrical functions were first compiled, agreement in regard to the unit of angle appears to be as far off as ever. The trigonometrical tables most recently published use various different units.

Historically, the most important system, of course, is the sexagesimal, and to this system many computers, such as astronomers and surveyors, are tied, because of their records and the graduations of their instruments. Such a restriction, however, does not obtain in the case of an increasing number of people who have to perform calculations involving angles, and for them the question is purely one of utility. Unfortunately, owing to non-agreement as to the fundamental unit, the application of decimalisation has led to the introduction of three different systems of units, and the publication of corresponding tables. In one case the whole circumference of the circle has been taken as the unit to be decimally subdivided; in another, the quadrant; and in a third, the existing degree which is a ninetieth of the quadrant. As regards the first two of these systems, although each of them—and more especially the second—has certain advantages, it may be doubted whether these are sufficiently great to justify that entire departure from the old units which the systems entail. On the other hand the decimal subdivision of the sexagesimal degree presents all the advantages of a decimal system, while conserving the existing unit. Many existing tables in the sexagesimal system already take six minutes as the advance of their argument, thus dividing the degree decimally. That some decimal system will ultimately triumph cannot be doubted, but this consummation, so much to be desired, may be unduly delayed by the conflicting claims of rival systems; and surely it would be wise in a case of this kind, when the obstacles to be overcome are so formidable, to take the line of least resistance by retaining the existing unit.

For certain special purposes it is important to have tables in which the unit of angle is the radian, and surprise may be expressed that only quite recently has a beginning been made with their construction. At present, so far as the writer is aware, nothing like comprehensive tables of circular functions to this unit is available. A small table has been included in Dr Knott's collection of mathematical tables, and a table of sines and cosines is given by Becker and Van Orstrand.

IV. Tables which at present are Rare or Wanting

Under this heading fall the radian tables just spoken of. To construct such tables probably little beyond compilation would be necessary. Whether it is desirable that a complete table should now be available, and if so, what form it should take as regards details, are questions on which this meeting might usefully express an opinion.

Another table which is somewhat rare is a table of logarithms of reciprocals, or 'co-logarithms,' as they are sometimes termed. This is the more remarkable because collections, including tables of logarithmic secants and cosecants—which are but the reciprocals of cosines and sines—rarely possess the analogous table in the case of numbers, that is, rarely possess a table of co-logarithms. It is quite reasonable to argue that it is not worth while to include any table merely for the purpose of saving the computer a subtraction, but it is hardly logical to afford him this relief in certain cases and to deny it in others.

Finally, it may be urged that there is great need of a convenient and comprehensive table of square roots. Such a table might be conveniently arranged in the manner of 'Chambers's' well-known seven-figure logarithmic tables, which have the first four figures of the argument placed at the side of the page and the fifth figure at the top, while convenient tables of proportional parts are provided for the remaining figures. The two operations of squaring and of extracting the square root are of such frequent occurrence that it should be unnecessary to labour the point of the need for such a table to facilitate the labours of the computer; yet, despite the importance of the matter, no table appears to exist. At present any one who requires to extract the square root of a seven- or eight-figure number, correct to the same number of figures, is compelled to use logarithms or some other equally round-about method. If only one set of tables, either squares or square roots, were to be provided instead of both—which of course would serve the double purpose, although less conveniently—it may be suggested that the table should be one of square roots rather than of squares. In the first place multiplying machines are now extensively used, and squares can readily be obtained by their aid, but not square roots. In the second place, a table of square roots, because it has twice as many entries as a table of squares—owing to the necessity for its being duplicated to take into account the effect of a shift in the decimal point—has smaller first differences than a table of squares for the same range of arguments: and the reduction in the values of the first differences is a distinct advantage.

V. List of Mathematical Tables referred to and Illustrated

In each case care has been taken to make the reproductions exactly full size.

Figure

1. *Four-Place Tables of Logarithms and Trigonometric Functions.* —E. V. Huntington. Published by The Harvard Co-operative Society. (Spon, London.)

   Five-Figure Logarithmic and Other Tables.—A. M'Aulay. (Macmillan, London.)

2. *Smithsonian Mathematical Tables. Hyperbolic Functions.*— G. F. Becker and C. E. van Orstrand. Published by the Smithsonian Institute.

   A New Table of Seven-Place Logarithms of all Numbers from 20,000 *to* 200,000.—E. Sang. (Layton, London, 1871.)

3. *Four-Figure Mathematical Tables.* (New and Enlarged Edition.)—C. G. Knott. (Chambers, Edinburgh.)

   Chambers's Mathematical Tables. (New Edition, 1893.) (Chambers, Edinburgh.)

4. *Logarithmic-Trigonometrical Tables.* (Vol. II.)—J. Bauschinger and J. Peters. (Published by W. Engelmann, Leipzig, 1911.)

5. *Tables of Logarithms and Anti-Logarithms to Five Places.*— E. Erskine Scott. (Layton, London.)

6. *Douglas's 2-colour Logs., Anti-logs., and Mathematical Tables.* (Simpkin, Marshall, London.)

7. *Five-Figure Tables of Mathematical Functions.*—J. B. Dale. (Arnold, London.)

8. *A.B.C. Five-Figure Logarithms.*—C. J. Woodward. (Second Edition.) (Spon, London.)

NOTE ON 'CRITICAL' TABLES

T. C. Hudson, B.A., F.R.A.S., H.M. *Nautical Almanac* Office, London

I should like to supplement Dr Milne's excellent paper by a few remarks on labour-saving devices known as 'critical' tables, that is tables arranged to show the required result, on mere inspection, to the nearest unit. Tables of this kind are of great assistance to a computer who, having to take many quantities from the same table, does not want to be continually performing mental interpolation. What he wants is the result to the nearest unit, and he wants it without uncertainty. Naturally therefore the series of the tabulated variable should correspond to the *half*-units in the result and not to the exact units themselves. A concrete example will make this clear. Let us suppose the computer is an actuary who wishes to convert a great number of three-figure decimals of a pound sterling into their equivalents in shillings and pence to the nearest penny. In this case much trouble would be saved by the use of a table of the form shown on the margin. This table shows at a mere glance that fifteen, sixteen, seventeen and eighteen thousandths of a pound are each equivalent to fourpence (to the nearest penny). The left-hand column consists of 'critical' quantities, or more precisely of the whole numbers next *larger* than those critical quantities which in this

Thou-sandths of a Pound.	Near-est Penny.
	0
3	
	1
7	
	2
11	
	3
15	
	4
19	
	5
23	
	6
28	
	7
32	
	etc.

case are surds. Thus 19 being really larger than the critical quantity gives 5 and not 4. This is indicated once for all by a south-east arrow.

In general, suppose the quantity to be tabulated is $y=F(x)$, where x is the argument. Let $x=\phi(y)$. Then the arguments to be tabulated are . . . . $\phi(y-\frac{1}{2})$, $\phi(y+\frac{1}{2})$, $\phi(y+\frac{3}{2})$, ..., and the results to be tabulated are . . . . y, $y+1$, . . ., and the table is the numerical expression of the following algebraic form:—

$$\begin{array}{ll} \cdots & \\ & \cdots \\ \phi(y-\frac{1}{2}) & \\ & y \\ \phi(y+\frac{1}{2}) & \\ & y+1 \\ \phi(y+\frac{3}{2}) & \\ & \cdots \\ \cdots & \end{array}$$

The method is capable of extension to the case of two variables, but usually it will be a case of one only.

If the tables are of considerable expansion, the first few figures which form the argument will change slowly. In this case it may be well to extrude them from the body of the table, either to the head of the page, opening, column, or other suitable place. Such extruded figures should be written or printed very boldly. As for the remaining figures of the argument, they should also be rather bold, so as to catch the eye easily. The results, which are given to the right on the half-line, need not be so bold, but should be no further to the right than is necessary.

ON A POSSIBLE ECONOMY OF ENTRIES IN TABLES OF LOGARITHMIC AND OTHER FUNCTIONS

J. E. A. Steggall, M.A., F.R.S.E., Professor of Mathematics in the University of St Andrews at University College, Dundee

Tables of functions are almost universally formed by entries based upon an arithmetical progression in the argument; and for many purposes, where in fact the extraction of a value is all that is needed, no better system can be imagined.

But inasmuch as very great accuracy could only, without further help, be attained at the cost of corresponding bulkiness in the tables, the use of first differences and proportional parts is also almost universal; the entries being still, as a rule, made for argument-values in arithmetical progression, and the increment in this progression being chosen roughly to suit the decimal or other system of numeration employed, and to secure that any calculation by interpolation of the functions for an argument lying between those of two adjacent entries shall be correct with at most a possible error of ±1 in the last figure.

It is fairly clear, without further explanation, that the interval or increment of argument for which the proportional-part method would give a result of uniform accuracy is by no means constant throughout the table; and that, in theory at least, all that we need for any function is a table containing only such entries as will enable the computer to calculate an intervening function with the same accuracy (subject to a ±1 error, instead of a ±·5 error) as the entries themselves possess, and that by a mere use of proportional parts.

It is therefore proposed to give a short note on this point, and to examine some of the practical ways that are open for reducing the present bulk of the tables without affecting their accuracy.

It must be borne in mind that any departure from the uniform increment of argument will involve somewhat more mental effort, and it must therefore be left for computers, familiar with and accustomed to the regular use of tables, and willing to experiment with the proposed new method of entry, to assess the magnitude of this effort and to decide whether the game is worth the candle.

In interpolating from values of $f(x)$ tabulated at intervals h we use the formula

$$f(a+x) \doteqdot f(a)+x\{f(a+h)-f(a)\}/h$$

where $x < h$, in place of the real value

$$f(a+x)=f(a)+xf'(a)+\tfrac{1}{2}x^2f''(a+\theta h),$$

and the error is, by Taylor's theorem

$$\begin{gathered} x\{f'(a)+\tfrac{1}{2}hf''(a)+\tfrac{1}{6}h^2f'''(a)+\ \ldots\} \\ -x\{f'(a)+\tfrac{1}{2}xf''(a)+\tfrac{1}{6}x^2f'''(a)+\ \ldots\} \end{gathered}$$

or, since x and h are small, very nearly $\frac{1}{2}x(h-x)f''(a)$, which is greatest when $x=\frac{1}{2}h$, in which case it is $h^2f''(a)/8$.

We have therefore to find the greatest value of h for which this error does not exceed a given small quantity ϵ: and the greatest interval that may be left between the entry $f(a)$ and the next entry $f(a+h)$ is given by $h^2=8\epsilon/f''(a)$.

This is the whole theory, and we now proceed to some examples.

Example 1.—A table of squares. Here $f''(a)=2$, and if a table be constructed, as by Barlow, giving the squares of all integers up to a certain point any interpolated square is certainly correct to within a maximum error of ·25 in excess, for in this case $h=1$ and therefore $\epsilon=·25$.

Example 2.—A table of cubes. Here $f''(a)=6a$ and the greatest error from interpolation in Barlow's tables is $3a/4$, which while numerically large is relatively small.

Example 3.—A table of square roots. Here $f''(a) = -1/4a^{\frac{3}{2}}$, and if the table runs, say, from $a=100$, to $a=1000$ at unit interval the maximum error varies from $-1/32000$ to $-1/1000000$ approximately. This interval of a unit is therefore needlessly small unless roots are required to an accuracy much closer than the arguments admit. Supposing then that roots are required to three decimal places the greatest interval h of entry admissible, ϵ being $\cdot0005$, is given by

$$h^2 = 32000 \times \cdot0005$$

at $a=100$, the beginning of the table, to

$$h^2 = 1000000 \times \cdot0005$$

at $a=1000$, the end of the table. Thus h increases from 4 to 22·4.

Now in practice the intervals must be integral, and even then all integers are not equally useful; our next step then is to ascertain and tabulate the values of a at which each integral difference from 4 to 22 appears. The formula is easily seen to be $2a = 25h(2h)^{\frac{1}{3}}$, where $h=4, 5, 6, \ldots 22, [23]$. The table is

h	a	h	a
4	100·0	14	531·5
5	134·6	15	582·6
6	171·7	16	635·0
7	210·9	17	688·5
8	251·9	18	743·0
9	293·6	19	798·4
10	339·3	20	855·0
11	385·3	21	912·5
12	432·6	22	970·8
13	481·3	[23]	1030·1

We have next to take sets of integers in arithmetical progression such that the first with a difference 4 just covers the interval from 100 to 134·6, the second with a difference 5 carries us on to 171·7, the third with a difference 6 to 210·9, and so on.

It is clear that the least number of entries is 84, thus reached:—

Difference	Entries	No. of Entries
4	100,104,108, ... ,136	10
5	141,146, ... ,176	8
6	182,188, ... ,212	6
7	219,226, ... ,254	6
8	262,270, ... ,294	5
9	303,312, ... ,348	6
10	358,368, ... ,388	4
11	399,410, ... ,443	5
12	455,467, ... ,491	4
13	504,517, ... ,543	4
14	557,571, ... ,585	3
15	600,615, ... ,645	4
16	661,677, ... ,693	3
17	710,727, ... ,744	3
18	762,780, ... ,816	4
19	835,854, ... ,873	3
20	893, ... ,913	2
21	934,955, ... ,976	3
22	998	1
		84

These 84 entries, in place of 900, are sufficient to give us the square root of all numbers less than 1000 correct to three decimal places using the ordinary simple interpolation of proportional parts.

But these entries are not yet well grouped for the computer, who would be worried not only with the unrelated points at which the argument-difference changes, but by the awkward differences. We proceed then to take some convenient stopping-places for each difference and to select only differences that lead to easy calculation. This is to some extent a matter for individual opinion, but an example may be given. Suppose that we resolve to deal only in complete hundreds, we then have entries

100	to	200	tabulated	with	difference	4
200	,,	300	,,	,,	,,	5
*300	,,	400	,,	,,	,,	8
400	,,	500	,,	,,	,,	10
*500	,,	600	,,	,,	,,	12
*600	,,	700	,,	,,	,,	14
*700	,,	800	,,	,,	,,	15
*800	,,	900	,,	,,	,,	18
900	,,	1000	,,	,,	,,	20

But the groups marked with a star do not include an integral number of differences, and we have therefore to recast the table thus

Entries	100	to	200	tabulated	with	difference	4
,,	200	,,	300	,,	,,	,,	5
,,	300	,,	500	,,	,,	,,	8
,,	500	,,	800	,,	,,	,,	12
,,	800	,,	1998	,,	,,	,,	18

This means 26+20+25+25+11+2 or 109 entries.

But little is lost and much is gained by restricting the differences to 4, 5, 10, in which case we have

Entries	100	to	200	tabulated	with	difference	4
,,	200	,,	400	,,	,,	,,	5
,,	400	,,	1000	,,	,,	,,	10

giving 26+40+60, or 126 entries, being under one-seventh the number usually employed.

This case has been worked at some length to illustrate the procedure. The next examples are interesting on account of certain special forms of difference or of result.

Example 4.—A table of logarithms. This case is interesting because of the ease with which the theoretically least number of entries can be calculated. Here $f''(a)=-1/a^2$, and $h^2=8\epsilon a^2$ for the Napierian system, and$=8\epsilon a^2/\mu$ for that of Briggs where μ is the usual constant of conversion ·434294 approximately. In the latter and more practical case a four-figure, a five-figure, a seven-figure, and an eight-figure table will be considered.

(A) In the four-figure table $\epsilon=\cdot 00005$, and $20000\ \mu h^2=8a^2$, whence $h=\cdot 030348a$: and as a passes from 100 to 1000, h changes from 3·03 to 30·3. It is clear that the theoretical number of entries is given by $x+1$, where $(1+h/a)^x=10$, and $x=1/\log(1\cdot 030348)=1/\cdot 012983=77$, and the number of entries is in theory 78. The table similar to that already used is

h	*a*	*h*	*a*	*h*	*a*
3	100·0	13	428·4	23	757·9
4	131·8	14	461·4	24	790·8
5	164·8	15	494·3	25	823·8
6	197·7	16	527·2	26	856·8
7	230·7	17	560·2	27	889·7
8	263·6	18	593·2	28	922·6
9	296·6	19	626·1	29	955·6
10	329·5	20	659·0	30	988·5
11	362·5	21	692·0	31	1018·5
12	395·4	22	725·0		

Now choosing the entries so that no discontinuity is introduced in passing from one increment-group to the next, we have the following shortest practical table :—

Difference.	Entries.		No. of Entries.	Difference.	Entries.		No. of Entries.
3	100 to	133	12	17	570 to	604	2
4	133	165	8	18	604	640	2
5	165	200	7	19	640	678	2
6	200	236	6	20	678	698	1
7	236	264	4	21	698	740	2
8	264	304	5	22	740	762	1
9	304	331	3	23	762	808	2
10	331	371	4	24	808	832	1
11	371	404	3	25	832	857	1
12	404	440	3	26	857	909	2
13	440	466	2	27	909	936	1
14	466	508	3	28	936	964	1
15	508	538	2	29	964	993	1
16	538	570	2	30	993	1023	0

The total number of entries here is 83, as against 900, but the changing places for differences are rather awkward. If we take 3 from 100 to 160, 4 from 160 to 200, 5 from 200 to 400, 10 from 400 to 700, and 20 from 700 to 1000, we need $21+10+40+30+15$ or 116 entries, compared with the ordinary 900.

(B) In the five-figure table $\epsilon=\cdot000005$ and $h=\cdot009583a$, and, as a passes from 100 to 1000, h changes from ·958 to 9·58: the number of entries required being the integer next above x where

$$(1+\cdot009583)^x=10$$

whence $x=240$, and our former table becomes

h	a	h	a
1	104·3	6	626·0
2	208·7	7	730·4
3	313·0	8	834·7
4	417·4	9	939·1
5	521·7	10	1043·4

Here we must either take the occasional risk of an extra half unit in our error or make a special case for numbers from 100 to 105: in fact, the divisions happen rather awkwardly; although if we had been satisfied with an accuracy within 2 (instead of 1) in the last figure we should have had a much shortened table. However if we take our numbers at intervals of 1 from 100 to 300, of 2 from 300 to 400, of 4 from 400 to 600, and of 5 the rest of the way, we should require 201+50+50+80, or 381 entries to give us all logarithms correct to five figures.

(C) In the seven-place table, that ordinarily used, $\epsilon = \cdot 00000005$ and $h = \cdot 0009583a$: thus as a changes from 10000 to 100000, h passes from 9·58 to 95·83. In this case we should have a very useful table, even although the 4 per cent. involved in the difference between 9·58 and 10·000 must sometimes cause an extra half unit of error in the last figure, by tabulating as follows:—

log 10000	to log 20000	at intervals of	10
,, 20000 ,,	,, 40000	,,	20
,, 40000 ,,	,, 60000	,,	40
,, 60000 ,,	,, 100000	,,	50

making a total of 2301 entries in place of the usual 90001: in other words, a book of 5 pages will do in place of one of 180, as far, that is to say, as the mere entries go. With the tabulation of differences the gain would not be quite as great, but in any case ten pages would carry all that would be required.

(D) The case of the eight-place table follows to some extent that of the four-place. Here $h = \cdot 00030348a$, and in such a table the theoretical minimum of entries is

$$1/\log(1 \cdot 00030348)$$

or 7594 in a table starting at 10^n and going to 10^{n+1}. The spacing of the entries of course varies with n, as in the other cases; the number of entries is fixed. The table under head A is sufficiently exact for our purpose, and it is clear that a difference 3 might run in the numbers from 10000 to 16000, 4 from 16000 to 20000, 5 from 20000 to 40000, 10 from 40000 to 70000, and 20 from 70000 upwards. The number of entries required for an eight-figure table would then be 2001+1000+4000+3000+1500 or 11501, that is to say, about one-ninth of the number used in the present seven-figure table.

The next example (5) will be a table of natural sines, the maximum error being $-\frac{1}{8}h^2 \sin \alpha$. If four figures are desired we must have $\frac{1}{8}h^2 \sin \alpha \ngtr \cdot 00005 \times 8$, giving $50h = \sqrt{\operatorname{cosec} \alpha}$, and the least value of h is rather less than one degree. Thus a sine, or cosine, table tabulated for every degree to four decimal places is sufficient to yield the same accuracy as the ordinary four-figure tables with entries at every 10 minutes of arc. It is plain that the portability of tables constructed on these principles goes some way to make up for the slightly longer work of interpolation demanded, but when one takes into account the many cases where an ordinary table is too minutely spaced for the immediate requirements one is tempted to believe that for the general work of the chemist, the physicist and the engineer, the modifications here proposed would far outweigh the disadvantage of (1) varying spaces, (2) increased interpolative calculation.

The last example (6) will illustrate the somewhat greater labour involved in settling the values of h when $f''(\alpha)$ is not quite so simple in form. Considering the table of logarithmic tangents we have $+(\alpha) \equiv \log \tan \alpha$, $f''(\alpha) = \cot^2\alpha - \tan^2\alpha$, and $\mu h^2(\cot^2\alpha - \tan^2\alpha) = 8\epsilon$. We may take α as $\ngtr \frac{\pi}{4}$ on account of the symmetry about this point; and for a four-figure table, h being expressed in circular measure, we have

$$h\sqrt{(\cot^2\alpha - \tan^2\alpha)} = \cdot 03035;$$

whence we have the following table :—

a	h	a	h
5°	9′	25°	50′
10°	18′	30°	1° 4′
15°	28′	35°	1° 24′
20°	38′	40°	2° 3′

For a seven-figure table

$$h\sqrt{(\cot^2\alpha-\tan^2\alpha)}=\cdot0009583;$$

whence we have the following table :—

a	h	a	h
5°	0′·288	25°	1′·573
10°	0′·581	30°	2′·016
15°	0′·884	35°	2′·646
20°	1′·209	40°	3′·892

From which it appears that the ordinary seven-figure tables for logarithmic tangents cannot be relied upon unless the angle lies between 16° and 74°, as is, of course, well known.

A small logarithmic five-place table was published on the principles shown many years ago, but the extensions to seven or eight places and to trigonometrical and other functions have not yet been printed.

THE GRAPHICAL TREATMENT OF SOME CRYSTALLOGRAPHIC PROBLEMS

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Few students of the Natural Sciences owe a greater debt to the genius of John Napier than do those who occupy themselves with the measurement and description of the manifold forms of crystals. That this should be so, is a striking illustration of the far-reaching applicability of Napier's discoveries, for he had been in his grave for more than half a century before the science of Crystallography was born. This event is commonly conceded to have taken place in 1669, when the learned Danish physician, Nicolaus Steno, published at Florence the work which appeared in English dress in 1671 under the title *Prodromus to a Dissertation concerning Solids naturally contained within Solids.* In this little book Steno established the important truth that the angles between the faces of crystals of the same substance, such as quartz or hematite, were always the same, in spite of great variations in the size and shape of the faces. The new science remained, however, but a puny infant for upwards of a hundred years, and it was only in the last quarter of the eighteenth century, under the fostering care of de Romé de l'Isle (*Essai de Cristallographie,* 1772), and especially of the Abbé Haüy (*Essai d'une théorie sur la structure des crystaux,* 1784), that it developed a vigorous growth, and became ready to take its place in the circle of the sciences. No sooner was it realised that the angles between the faces of crystals, far from being independent, exhibited interesting relations which were capable of simple mathematical expression, than crystallographers found in the inventions and discoveries of Napier powerful tools ready forged to their hand.

It is true that the simplest geometrical methods serve to make clear the relations which prevail in many crystals, such as the cube, the regular octahedron and the rhombic dodecahedron. It is easy, for instance, to show that the cosine of the angle between the face of a cube and the face of the octahedron which truncates the cube corner and makes equal angles with the cube faces, is equal to $1/\sqrt{3}$ whence the angle is found to be 54° 44′. The methods employed by Haüy were of this simple geometrical character, and only occasionally did they lead him to expressions for whose solution he was glad to call in the aid of the logarithms of the trigonometrical functions. Such methods have, however, grave limitations, and when the attempt is made to solve by these means alone the difficult problems presented by complicated crystals of a low degree of symmetry, and which cannot be referred to rectangular axes, the way is found to be long and tedious, and the resulting formulæ inelegant and troublesome to compute. It is indeed probable that the unfortunate predilection of certain well-known crystallographers of the middle of last century for these cumbrous processes is responsible for crowding their pages with unnecessarily long and repellent formulæ, and incidentally for frightening away chemists and geologists from a study which can afford them invaluable assistance in their own researches.

A step along a new and more promising road was taken early in the nineteenth century, when it was discovered that spherical trigonometry could be utilised with advantage in the solution of crystal problems. Thus in the Preface to his *Familiar Introduction to Crystallography*, 1823, H. J. Brooke tells us that on the advice of Lévy he had used spherical trigonometry in the portion of his book which dealt with crystal calculation. About the same time F. E. Neumann (*Beiträge zur Krystallonomie*, 1823) pointed out that crystals can be conveniently represented by drawing normals to their faces from an origin within the crystal and producing these normals till they meet the surface of a sphere surrounding the crystal and having the origin as centre. This mode of treatment gives the most general representation of a crystal possible, for all unessential variations due to the relative sizes and shapes of the faces are eliminated, each face being represented by a point on the sphere. Further, all faces which on the crystal belong to the same zone, are on the sphere represented by points which lie on the same great circle. The surface of the sphere is mapped out by these great circles into a network of spherical triangles which in many cases are right-angled. The greater number of crystallographic problems resolve themselves into the solution of these triangles from data supplied by measurement, and here the well-known rules devised by Napier find their application. It is obvious that it will usually be convenient to represent the sphere by projecting it on a plane, and this can be accomplished in several different ways. The two which have proved most useful in practice are the stereographic and the gnomonic projections. Both were well known in Napier's time, and are now used to the practical exclusion of all others. The former is best adapted for purposes of computation, the latter lends itself particularly well to graphical methods of calculation.

No one realised more fully than W. H. Miller the advantages of Neumann's mode of treatment. In his writings on Crystallography he adopted it whole-heartedly, and combining it with a suggestion due to W. Whewell, he devised the notation associated with his name. His *Treatise on Crystallography*, published in 1839, written entirely from this standpoint, contains a number of elegant expressions, most of them original, and forms the foundation on which all subsequent developments have been based. In spite of its superiority to all others, Miller's notation spread but slowly, and though its inherent merits have won recognition in the end, it can only be said to have finally distanced its rivals in the last few years.

We see then that, thanks largely to Miller's influence, descriptive Crystallography is now in the main an affair of the solution of spherical triangles. When these triangles are right-angled triangles, as is so frequently the case, the risk of error in calculation is but small. When, however, the problem involves the solution of many oblique-angled triangles mistakes are more liable to occur, and it is found advantageous to check the results by graphical methods. This is now often effected by means of stereographic nets printed on tracing paper, on which a system of great and small circles is laid down once for all, usually at 2° intervals. It is, however, probable that the ingenious methods associated with the name of Mons. Maurice d'Ocagne might be employed with advantage in this connection. Indeed for the approximate solution of certain crystallographic problems simple Nomograms have been in use for some time, as the following examples will show.

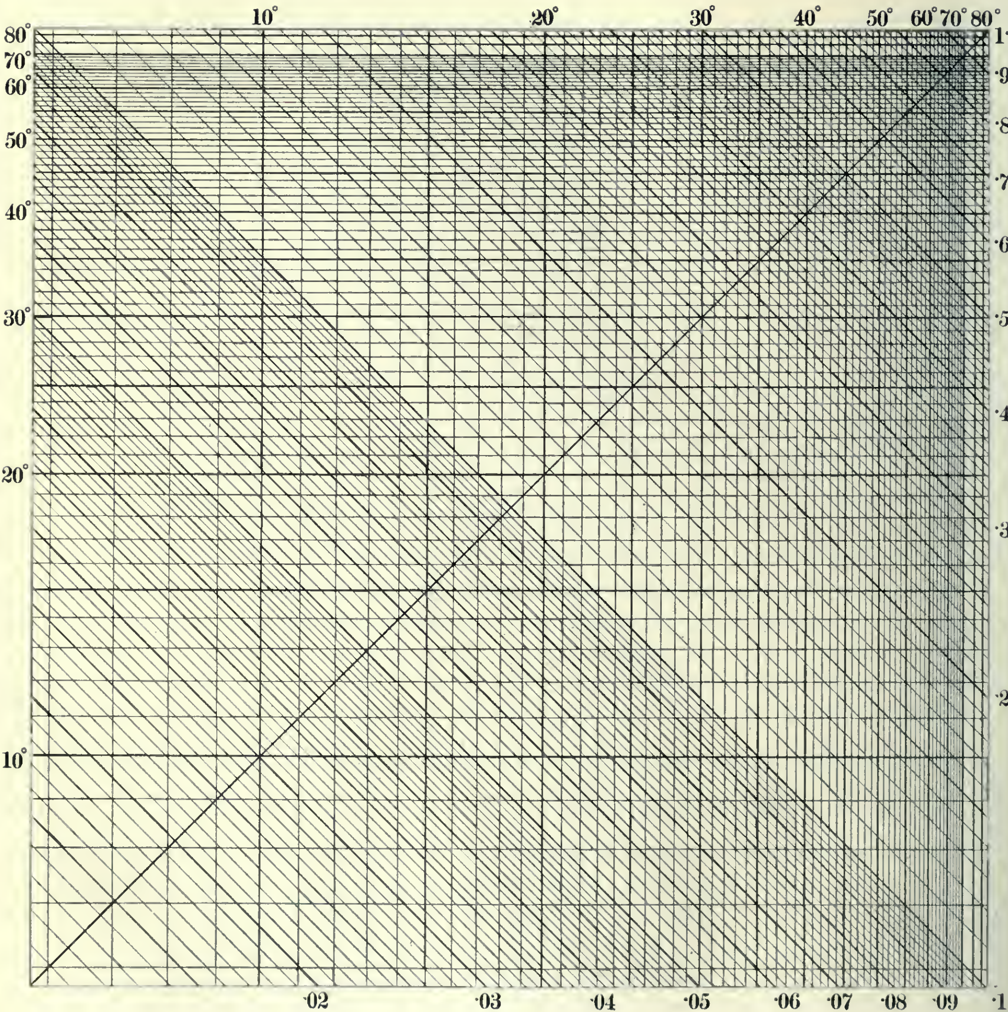
Among the relations utilised by Miller that which connects the angles between four faces which belong to the same zone with the indices of the faces in Miller's notation, is at once one of the most interesting and most useful. It is generally expressed as follows :—

If P, Q, R, S are four such faces, then $\frac{\sin PQ \sin SR}{\sin PR \sin SQ}=m$,
where m is a simple fraction $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, etc.

If the angle PS is a right angle this relation reduces to the simpler form $\frac{\tan PQ}{\tan PR}=m$.

For the rapid determination of m in the more general

PLATE XIV



10°
20°
30°
40°
50°
60°
70°
80°
80°
70°
60°
50°
40°
30°
20°
10°
1·0
·9
·8
·7
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·5
·4
·3
·2
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·03
·04
·05
·06
·07
·08
·09
·1

form of the expression, the writer has found it advantageous to use a diagram closely resembling L. Lalanne's well-known abacus for multiplication and division (*Description et usage de l'abaque*, etc., 1845; *Explanation and use of the Abacus or French Universal Reckoner*, London, 1846).

This nomogram (Plate XIV) is constructed by marking off on two rectangular axes distances proportional to the logarithmic sines of the angles from 5° 45′ to 90°. It will suffice if every degree is shown from 6° to 70°. Through these points lines are drawn parallel to the axes, and finally diagonals are ruled in corresponding to the values of the products of the sines from 0·01 to 1·0. To evaluate the expression $\frac{\sin A \sin B}{\sin C \sin D}$ we proceed as follows:—The diagonal through the point E (Fig. 1) gives us the value of the product sin A sin B. In order to divide this product by sin C the diagonal through E is followed till it meets at F the abscissa drawn through C. The result may then be read off at G, but as it is not required, we proceed at once to divide further by sin D. To accomplish this the diagonal from G is followed till it meets at H the abscissa drawn from D. The continuation of the ordinate through H gives at K the value of m.

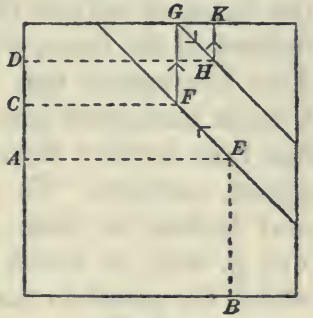

Fig. 1.

A still easier plan, which is often satisfactory, is to find the values of the numerator and denominator separately; inspection will then generally decide the value of m. Thus in the following example

$$\frac{\sin 18° \, 25' \sin 34° \, 43'}{\sin 29° \, 54' \sin 46° \, 12'} = m$$

the numerator and denominator are found to be approximately equal to 0·18 and 0·36 respectively. It follows at once that $m = \frac{1}{2}$. In either case the result is arrived at much more quickly than if worked out by the aid of a table of 4-figure logarithms. This nomogram is also capable of doing good service in the rapid evaluation of a number of expressions involving sines frequently met with in Crystallography and Optics. Further, it enables us to read off the value of $\sin^2\theta$ and of $\cos^2\theta$ to within a unit or two of the third significant figure.[1]

For the solution of the simple case where the expression involving sines is reduced to a ratio of two tangents, an ingenious nomogram devised by Dr G. F. Herbert Smith is available. ('The Construction and Use of the Moriogram,' *Mineralogical Magazine*, 1904, vol. xiv, p. 49.) By its aid we can quickly determine the possible values of the angle PQ corresponding to a given value of PR and the indices of the faces, or we can find the value of m, and therefore of the indices of the faces, when the angles PQ and PR are known. This nomogram can also be applied to the general case, though somewhat less conveniently. The latter application has been further discussed by Dr Herbert Smith in a recent paper, entitled 'The Graphical Determination of Angles and Indices in Zones' (*Mineralogical Magazine*, 1913, vol. xvi, p. 326), in which he has described two new methods by which the problem may be solved, and has given the necessary diagrams.

The simple case can also be dealt with very easily by the aid of a nomogram constructed like the abacus of sines described above, logarithmic tangents being plotted instead of the logarithmic sines. When this is done, the radiating lines and curves of Dr Herbert Smith's diagram become two systems of straight lines parallel to the diagonals of the square. The writer has found recently that a still more convenient arrangement is arrived at if this tangent diagram

[1] A number of diagrams especially designed for solving equations of the type $\sin i = n \sin r$ have been devised by Dr F. E. Wright, vide *The Methods of Petrographic-Microscopic Research*, Washington, 1911, and *Amer. Journ. Science*, 1913, vol. 36, p. 509, see also *The Mineralogical Magazine*, 1913, vol. xvi, p. 236.

PLATE XV

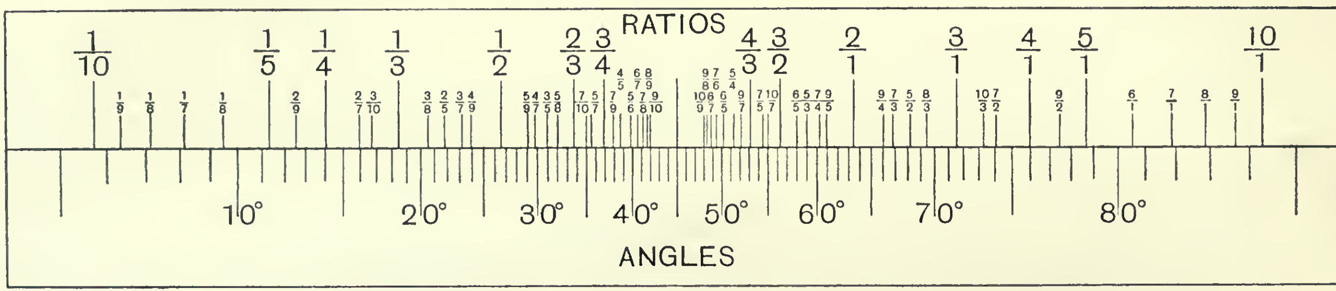

Fig. 1.

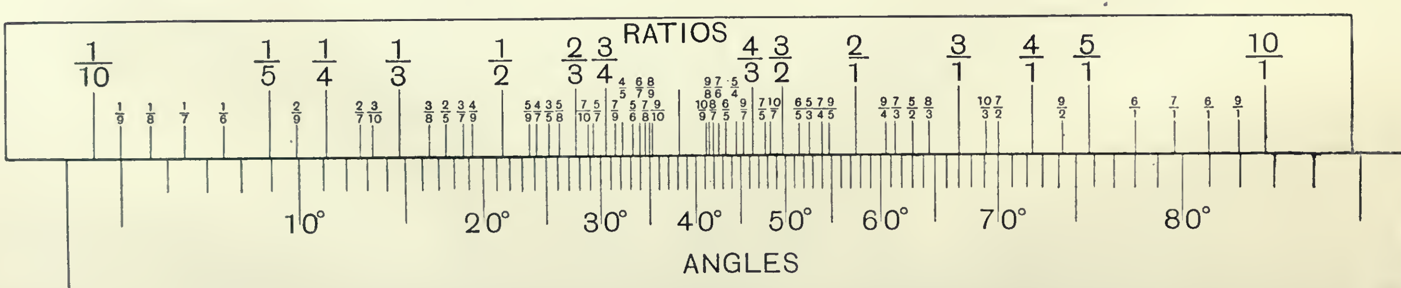

Fig. 2.

is transformed into a slide-rule constructed as follows :— A scale of logarithmic tangents from 5° to 85° is drawn at the edge of a strip of stout card. On another similar card distances are marked off proportional to the logarithms of the fractions $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, etc., and also to the logarithms of their reciprocals 3, 2, 1·5, etc., graduations being made for all fractions of which the numerator does not exceed 9 and the denominator 10. By placing the two graduated edges in contact we are in a position to read off all the indices in any zone of 90° in which the angles are known, or conversely, we can find the angles corresponding to the indices and a given parametral angle.

Two examples will make the mode of use quite clear.

In the cubic system the faces 100, 010 are at right angles, and the parametral face 110 is at 45° to each of them. If now we place the graduation corresponding to 45° on the scale of angles, opposite the central mark on the scale of ratios, we can at once read off the angles which the faces 310, 210, 021 make with the face 100, for the tangents of these angles are 1/3, 1/2, and 2, and the corresponding angles are 18° 26′, 26° 34′, and 63° 26′ respectively. See Plate xv, Fig. 1.

As a second example we may choose the mineral anglesite, sulphate of lead. Crystals of this substance often exhibit two faces A and B at right angles to one another, which receive the indices 100 and 010 respectively, and between these faces a number of others have been observed, namely, M, N, λ, P, *i*, Q, *m*, T, U, *h*, δ, V, *n*, κ, W. The angles which these faces make with the face A are shown in the following table :—

M	11° 6′	Q	30°30′	δ	49°40′
N	14 40	*m*	38 $8\frac{1}{4}$	V	51 29
λ	21 26	T	41 54	*n*	57 $30\frac{1}{2}$
P	24 10	U	45 16	κ	67 0
i	27 38	*h*	46 $18\frac{1}{2}$	W	70 0

We first select a parametral plane to which to assign the indices 110. Let us choose *m* as this plane. If now we set the angle Am=38° 8¼′ opposite the central mark of the scale of ratios (see Plate xv, Fig. 2) we find opposite each of the other tabulated angles the ratio which gives us the corresponding indices, thus opposite 11° 6′ we find the ratio 1/4, and therefore M is 410, similarly N=310, λ=210, P=740, i=320, Q=430, T=780, U=790, h=340, δ=230, V=580, n=120, κ=130, W=270. Conversely, if we are given the indices we can at once read off the angle which each of the given faces makes with A or B.

For demonstration purposes the author has constructed a slide-rule 50 inches long, graduated as described, and by its aid the angles corresponding to each of the faces mentioned above were rapidly read off, before the tabulated angles were calculated by means of a 5-figure table. The values were as follows :

face	read	calculated	face	read	calculated	face	read	calculated
M (410)	11° 7′	11° 6′	Q (430)	30°30′	30°30′	δ (230)	49°41′	49°40′
N (310)	14 41	14 40	m (110)	*38 8¼	...	V (580)	51 30	51 29
λ (210)	21 26	21 26	T (780)	41 58	41 54	n (120)	57 32	57 30½
P (740)	24 9	24 10	U (790)	45 18	45 16	κ (130)	67 1	67 0
i (320)	27 38	27 38	h (340)	46 20	46 18½	W (270)	70 1	70 0

It will be seen that the angles read off from the rule agree very closely with those afterwards calculated, the maximum difference being 4 minutes. Such a degree of accuracy is amply sufficient for many purposes.

In conclusion it may be pointed out that, when engaged in assigning indices to a large number of faces lying in such a zone, it is often desirable to try tentatively the effect of choosing various faces as parametral planes, in order to see which selection gives on the whole the simplest indices to the rest. The arrangement described lends itself very conveniently to this operation, for all that is necessary is to bring the angle corresponding to any face to the central graduation of the scale of ratios, when the indices corresponding to the other faces can be read off direct.

A METHOD OF COMPUTING LOGARITHMS BY SIMPLE ADDITION

WILLIAM SCHOOLING, F.R.A.S.

Since simple addition is the easiest arithmetical process many advantages attach to methods which enable it to be employed for extensive computations. It is the purpose of this paper to show that, by means of a peculiar geometrical progression, long series of logarithms and anti-logs can be obtained by addition.

The logarithms of the successive terms of any geometrical progression can be obtained by taking the logarithm of the first term and repeatedly adding the logarithm of the common ratio.

Such a series of logarithms would be of little use unless we had a ready means of finding the corresponding natural numbers. This is available if we employ a geometrical progression with the particular characteristic that the sum of any two successive terms equals the next term.

If a is the first term of such a progression and ϕ the common ratio then

(1)
$$a\phi^n + a\phi^{n+1} = a\phi^{n+2}$$
$$a + a\phi = a\phi^2$$
$$1 + \phi = \phi^2$$
$$\phi = (\sqrt{5}+1)/2 = 1{\cdot}618034$$
$$\text{or } \phi = (-\sqrt{5}+1)/2 = -{\cdot}618034$$

Also

(2)
$$a\phi^{n+2}+a\phi^{n+1}=a\phi^n$$
$$a\phi^2+a\phi=a$$
$$\phi^2+\phi=1$$
$$\phi=(\sqrt{5}-1)/2=\cdot618034$$
$$\text{or } \phi=(-\sqrt{5}-1)/2=-1\cdot618034$$

Hence
$$\phi=\pm1\cdot618\ 033\ 988\ 749\ 894\ 848+$$
$$\phi=\pm\ \cdot618\ 033\ 988\ 749\ 894\ 848+$$

I give a general example of this progression

n	$\log a\phi^n$.	$a\phi^n$.
0	$\log a$	$1a+0\ \phi=a(1+0\phi)=a$
1	$\log a+1\log\phi$	$0a+1a\phi=a(0+1\phi)=a\phi$
2	$\log a+2\log\phi$	$1a+1a\phi=a(1+1\phi)=a\phi^2$
3	$\log a+3\log\phi$	$1a+2a\phi=a(1+2\phi)=a\phi^3$
4	$\log a+4\log\phi$	$2a+3a\phi=a(2+3\phi)=a\phi^4$
5	$\log a+5\log\phi$	$3a+5a\phi=a(3+5\phi)=a\phi^5$

Thus the logarithms are obtained by repeated addition of the logarithm of the common ratio, and the corresponding natural numbers by adding the previous two consecutive terms.

In the following numerical examples $a=1$ and $a=4$. The logarithms are given to base ϕ and base 10

$$\log_{10}\phi=0\cdot208\ 987\ 640<0\cdot209$$
$$\log_{10}4=0\cdot602\ 059\ 991>0\cdot602$$
$$\log_{\phi}4=2\cdot880\ 840\ 180<2\cdot881$$

TABLE I

n = Number	$\log_\phi N$	$\log_{10} N$
$1 = 1{\cdot}000$	0	0·000
$\phi = 1{\cdot}618$	1	0·209
$\phi^2 = 2{\cdot}618$	2	0·418
$\phi^3 = 4{\cdot}236$	3	0·627
$\phi^4 = 6{\cdot}854$	4	0·836
...	...	...
$4 = 4{\cdot}000$	2·881	0·602
$4\phi = 6{\cdot}472$	3·881	0·811
$4\phi^2 = 10{\cdot}472$	4·881	1·020
$4\phi^3 = 16{\cdot}944$	5·881	1·229
$4\phi^4 = 27{\cdot}416$	6·881	1·438

Three-figure logarithms suffice for illustrating the method, but its advantages are most marked when many places are required.

Such series as the above may be continued as far as we please, but though the first terms of any series or geometrical progression may be varied indefinitely the common ratio is always ϕ. Hence a series of numbers the logarithms of which differ by log ϕ constitute an additive series.

There is another way in which the ϕ progression may conveniently be used for calculating logarithms,

$\phi^n + \phi^{-n}$ is integral when n is even

$\phi^n - \phi^{-n}$,, ,, n ,, odd.

Also

$\phi^n + \phi^{-n}$ is an integral multiple of $\sqrt{5}$ when n is odd

$\phi^n - \phi^{-n}$,, ,, ,, $\sqrt{5}$,, n ,, even.

This is seen in Table II, giving the terms of the series F (or Fibonacci) and H. F_n is the nth term of the F series.

TABLE II.

n	F_n	H_n
1	$1=(\phi^1+\phi^{-1})/\sqrt{5}=\phi^1(1+\phi^{-2})/\sqrt{5}$	$1=\phi^1-\phi^{-1}=\phi^1(1-\phi^{-2})$
2	$1=(\phi^2-\phi^{-2})/\sqrt{5}=\phi^2(1-\phi^{-4})/\sqrt{5}$	$3=\phi^2+\phi^{-2}=\phi^2(1+\phi^{-4})$
3	$2=(\phi^3+\phi^{-3})/\sqrt{5}=\phi^3(1+\phi^{-6})/\sqrt{5}$	$4=\phi^3-\phi^{-3}=\phi^3(1-\phi^{-6})$
4	$3=(\phi^4-\phi^{-4})/\sqrt{5}=\phi^4(1-\phi^{-8})/\sqrt{5}$	$7=\phi^4+\phi^{-4}=\phi^4(1+\phi^{-8})$
5	$5=(\phi^5+\phi^{-5})/\sqrt{5}=\phi^5(1+\phi^{-10})/\sqrt{5}$	$11=\phi^5-\phi^{-5}=\phi^5(1-\phi^{-10})$
6	$8=(\phi^6-\phi^{-6})/\sqrt{5}=\phi^6(1-\phi^{-12})/\sqrt{5}$	$18=\phi^6+\phi^{-6}=\phi^6(1+\phi^{-12})$
7	$13=(\phi^7+\phi^{-7})/\sqrt{5}=\phi^7(1+\phi^{-14})/\sqrt{5}$	$29=\phi^7-\phi^{-7}=\phi^7(1-\phi^{-14})$
8	$21=(\phi^8-\phi^{-8})/\sqrt{5}=\phi^8(1-\phi^{-16})/\sqrt{5}$	$47=\phi^8+\phi^{-8}=\phi^8(1+\phi^{-16})$
9	$34=(\phi^9+\phi^{-9})/\sqrt{5}=\phi^9(1+\phi^{-18})/\sqrt{5}$	$76=\phi^9-\phi^{-0}=\phi^9(1-\phi^{-18})$
10	$55=(\phi^{10}-\phi^{-10})/\sqrt{5}=\phi^{10}(1-\phi^{-20})/\sqrt{5}$	$123=\phi^{10}+\phi^{-10}=\phi^{10}(1+\phi^{-20})$

If we know, to any base, $\log\phi$, $\log(1+\phi^{-n})$ and $\log(1-\phi^{-n})$ for even integral values of n we can find the logarithms of the terms of the F and H series. The natural numbers of these terms are obtained by the addition of the last two terms. The series include many prime numbers. For example, in H there are

3, 7, 11, 29, 47, 76(4×19,) 123(3×41), etc.

It is a well-known result of the Exponential Theorem that

$$\log_e(1+x)=x-x^2/2+x^3/3-x^4/4+$$
$$\log_e(1-x)=-(x+x^2/2+x^3/3+x^4/4+)$$

Put ϕ^{-n} for x and M for the modulus for common logarithms and we have

$$\log_{10}(1+\phi^{-n})=M\phi^{-n}-M\phi^{-2n}/2+M\phi^{-3n}/3-M\phi^{-4n}/4+$$
$$\log_{10}(1-\phi^{-n})=-(M\phi^{-n}+M\phi^{-2n}/2+M\phi^{-3n}/3+).$$

We saw (page 338) that $a\phi^n$ and $\log a\phi^n$ for values of n differing by unity can be formed by addition or subtraction. Therefore by commencing with M and $M\phi$ we can find other values of $M\phi^n$ by addition or subtraction. In calculating logs to base 10 we multiply ϕ by M and have no occasion to multiply by the modulus again. This is a great saving when computing logarithms to many places.

Take as an example $\log_{10} 15127$, which is the 20th term of the H series.

$$15127=\phi^{20}+\phi^{-20}=\phi^{20}(1+\phi^{-40})$$

$$\log_{10}(1+\phi^{-40})=M\phi^{-40}-M\phi^{-80}/2+M\phi^{-120}/3-.$$

The first significant figure of ϕ^{-120} is in the 26th decimal place. Hence to 20 places we have

$M\phi^{-40}$	=0·00000 00018 97923 49150
$-M\phi^{-80}\div 2$	=0·00000 00000 00000 00415
$\log_{10}(1+\phi^{-40})$	=0·00000 00018 97923 48735
$20\log_{10}\phi$	=4·17975 28049 99574 67539
$\log_{10}15127$	=4·17975 28068 97498 16274

The figures required are taken from Tables of $\log(1\pm\phi^{-n})$ and of $\log \phi^n$, many of the items in which are used repeatedly.

The terms of the F and H series and their products give a large number of logarithms of other numbers, both the numbers and the logs being found by addition. When a is any number

$$aF_n+aF_{n+1}=aF_{n+2}$$

$$aH_n+aH_{n+1}=aH_{n+2}.$$

Thus the numbers are found by adding the last two numbers to obtain a new term: and the logs by

$$\log a+\log F_n \text{ or } \log a+\log H_n.$$

We can also employ any other additive series of the Fibonacci type. If a is the first, b the second, and S_n the nth term

$$S_{n+2}=aF_n+bF_{n+1}$$

but $F_{n+1}=\phi F_n\pm\phi^{-n}$

$\therefore$ $S_{n+2}=F_n(a+b\phi)\pm b\phi^{-n}$

$a+b\phi=C$, a constant for the series

$$S_{n+2}=CF_n\pm b\phi^{-n}$$

$$=CF_n(1\pm b/CF_n\phi^n)$$

Where b is positive the sign is $+$ when n is even, and $-$ when n is odd.

The logs of b, C, F_n, and ϕ^n are all known. The expression $b/CF_n\phi^n$ is very small when n is large; consequently $\log(1\pm b/CF_n\phi^n)$ is readily obtained from tables of $\log\phi^{-n}$, $\log(1+\phi^{-n})$, and $\log(1-\phi^{-n})$ to any base. These tables can be computed by addition or subtraction, many of the items in them being required in connection with the F and H series.

If $a=3$, $b=1$, we have among the terms of the series the following prime numbers

n	S_{n+2}
6	$37=F_{19}\div 113=CF_6+\phi^{-6}$
8	$97 \quad =CF_8+\phi^{-8}$
9	$157 \quad =CF_9-\phi^{-9}$
10	$254=127\times 2 \quad =CF_{10}+\phi^{-10}$
11	$411=137\times 3 \quad =CF_{11}-\phi^{-11}$
13	$1076=269\times 4 \quad =CF_{13}-\phi^{-13}$

In this case the constant

$$C=3+\phi=\phi^3+\phi^{-2}=\phi^3(1+\phi^{-5})$$

For example

$$\begin{aligned}1076&=233(3+\phi)-{\cdot}00191938\\&=1076{\cdot}00191938-{\cdot}00191938=1076+\phi^{-13}-\phi^{-13}\\&=CF_{13}(1-1/CF_{13}\phi^{13})\\&=CF_{13}(1-{\cdot}00000178)\\&=1076{\cdot}001919\times{\cdot}99999822=1076.\end{aligned}$$

$\text{Log}\ 1076=\log C+\log F_{13}+\log(1-1/CF_{13}\phi^{13})$.

Four points are worth mention:

1. Almost any additive series of the Fibonacci type gives many primes and multiples of primes.

2. Frequent multiplication by the modulus is avoided, since it is not necessary to find $\log_e$ of prime numbers.

3. The progression enables Gaussian logs to be formed with great facility. Tables giving $\log(1 \pm 1/CF_n\phi^n)$, which is the Gaussian log of $\log(1/CF_n\phi^n)$, are needed for only a small range and can be easily computed to many places.

4. Almost the whole of the computation can be put in a form which makes each result depend on previous calculations and which lends itself to simple checks.

It is not now worth while to re-compute logarithmic tables, but had the ϕ progression been employed originally an enormous amount of labour would have been saved. It may be added that by the ϕ method, when some preliminary values for each series have been calculated, the work of finding logarithms and anti-logarithms to 100 places is only twice that of finding them to 50 places; four times that of computing them to 25 places; and ten times that of ascertaining them to 10 figures.

RECIPROCALS.

It may be of interest to point out that the ϕ progression is convenient in computing reciprocals by addition or subtraction.

If $$Aa=1,\ A\phi^n \times a\phi^{-n}=1.$$

We have seen (p. 338) that $A\phi^n$ and $a\phi^{-n}$ for all desired values of n, differing by unity, can be thus formed.

We can give A any value we please, and after finding $A\phi^n$, a, and $a\phi^{-n}$, we can form as many pairs of reciprocals as we like.

The successive multiples of ϕ cannot be expressed exactly in figures, but we can subtract from, or add to $A\phi^n$ a small quantity D, thus obtaining an exact number $A\phi^n \mp D$ the reciprocal of which is $a\phi^{-n} \pm d$.

If B and b are any two reciprocals

$$Bb=1=(B-D)(b+d)=1+Bd-bD-dD$$

$$d(B-D)=bD$$

$$d=\frac{bD}{B(1-bD)}=\frac{b^2D}{1-bD}$$

$$=b^2D+b^3D^2+b^4D^3+$$

If $(B+D)(b-d)=1$

$$d=b^2D-b^3D^2+b^4D^3-b^5D^4+$$

Since $Bb=1$, $Bb\times10^{-m}$ gives 1 in the mth place of decimals.

Hence $b^3D^2=10^{-m}$ when $D=B^{\frac{3}{2}}\times10^{-\frac{m}{2}}$

$b^4D^3=10^{-m}$,, $D=B^{\frac{4}{3}}\times10^{-\frac{m}{3}}$

$b^5D^4=10^{-m}$,, $D=B^{\frac{5}{4}}\times10^{-\frac{m}{4}}$.

If D is increased by the addition of E, 2E, 4E, etc. (where $E=10^{-k}$, say ·0001)

d becomes $b^2(D+E)+b^3(D+E)^2+b^4(D+E)^3+$
$b^2(D+2E)+b^3(D+2E)^2+b^4(D+2E)^3+$
$b^2(D+3E)+b^3(D+3E)^2+b^4(D+3E)^3+$.

The second differences are always $2E^2b^3$.

If therefore we have many pairs of reciprocals, easily found by means of the ϕ progression, we can form reciprocals of intermediate numbers by first and second differences.

When the third term, b^4D^3, in the value of d becomes significant we begin again on fresh values of $A\phi^n$ and $a\phi^{-n}$.

With a Difference Machine of simple design both reciprocals and logarithms could not only be computed automatically, but the results could be photographed as they were obtained. From the photograph a block of a page could be made from which tables could be printed. These would be free from errors due to either computers or compositors, and proof-reading would be unnecessary.

The ϕ progression has many applications in addition to those connected with logarithms, and in developing some of its features I have been greatly indebted for numerous valuable suggestions to my friend Mr Mark Barr.

HOW TO REDUCE TO A MINIMUM THE MEAN ERROR OF TABLES

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In the years which have passed since the memorable invention of logarithms, different systems have been devised for the arrangement and calculation of tables of logarithms, chiefly for the purpose of combining rapidity and convenience in use with a considerable degree of accuracy. Some of these systems will be discussed in the following remarks.

1. Firstly, we will consider the simple type of tables, which are intended for ordinary linear interpolation. Sometimes the first-differences are directly given, and, as a rule, tables of proportional parts are provided, while economy of space is effected by the double-entry arrangement (already used by J. Newton in the seventeenth century). For the sake of brevity this type of tables will in the following be referred to as 'Type A.' They are, even nowadays, preferred by many calculators, although other types (see below) undoubtedly surpass them as to convenience and rapidity.

Now let us consider accuracy. We may here suppose that the higher differences are unimportant, or the function nearly linear. It is here essential to give, not the maximal error, nor the probability of some special value of the error, but the mean-square of errors (which is the square of the mean-error, according to the general significance of this expression in the theory of probabilities). As unit of the error we will use the last decimal unit of the table-values. If we consider only the values directly printed, the mean-square of errors is known to be 1/12. If we suppose the interpolations distributed evenly along the whole table-interval, we find the mean-square of errors to be

$$1/18+1/12=5/36, \text{ or } 0{\cdot}1388 \ldots\ldots$$

If, however, only the points dividing the interval into 10 equal parts are considered, we find a slightly lower value,

$$\frac{403}{3000}, \text{ or } 0{\cdot}13433 \ldots\ldots.$$

It is not necessary to give here the exact law for the distribution of the errors, which is, furthermore, not very different from the 'normal' law of errors; the reader interested in this question should consult *De accuratione qua possit quantitas per tabulas determinari*, by Carolus Æmilius Mundt, Havniæ, 1842.

2. We will now consider another well-known type, which will hereafter be referred to as Type B. The ordinary double-entry table is here accompanied by a special auxiliary table, the separate horizontal lines of which correspond to the horizontal lines of the main table. By means of the values of the auxiliary table, which are simply 1, 2, 3 . . . 9-tenths of the mean-difference of the opposite part of the main-table, all the interpolations necessary are reduced to simple additions. This convenient form of table seems to have been used not earlier than in the nineteenth century, and the oldest tables,[1] as far as I know, are the following two: *Fünfstellige Logarithmen*, by A. M. Nell (1866), and

[1] Note by Editor.—Augustus de Morgan, in the Section 'Arithmetic and Algebra' of the first volume of *Mathematics* in the Library of Useful Knowledge (London, 1836), used this method of arranging Proportional Parts in his Four-Figure Tables of Logarithms and Antilogarithms. These tables were afterwards printed on two sides of one sheet of cardboard and published with 'A. de Morgan' printed at the foot. These sheets were in use in the early days of the Physical Laboratories in the Universities of both Edinburgh and Glasgow.

Logarithms and Antilogarithms, issued by the Institute of Actuaries in 1877 (4 figures). The arrangement used in some parts of *Tables trigonométriques décimales*, by Borda and Delambre (1801) is, however, essentially the same. Obviously some modifications of this type of table are possible; but the arrangement described here is the most natural, because the number 10 is the basis of the system of numeration.

Considering accuracy, we find the mean-square of the error to be

$$1/6 \text{ or } 0{\cdot}1666 \ldots\ldots$$

3. An increase of accuracy, without loss of convenience, has been accomplished by the appearance of *Fircifret Logaritmetabel*, by N. E. Lomholt, 1897 (the first edition). This author's object has been to do away with the great errors (exceeding 1·05 units), and furthermore to diminish the number of errors exceeding 0·5, as well as to diminish the 'average error.' He has not, as sometimes stated (for example, in the *Mathematical Encyclopædia*, both editions), reduced the average error to a minimum, and he has not thought it necessary to use a definite method, excluding entirely the personal element. Although, in my opinion at least, this is a drawback, I willingly acknowledge not only the real progres made, but also the general idea of improving the tables of Type B by means of new values both in the main table and in the auxiliary table.

4. In a Danish paper, published in *Nyt Tidsskrift for Matematik*, 1910, the present writer proposed to choose the 10+9 or 19 values of each horizontal line in such a manner that the sum of the square of the 100 resulting errors will be reduced to a minimum. It is hardly necessary to say anything here in defence of this proposal; but I will shortly set forth the method by which the 19 values may be found, using as an example the calculation of a single line (61) of the 4-figure table of logarithms.

Let us, for the present at least, accept the values of a 7-figure table as exact. As a starting-point we shall use a preliminary set of 19 values taken from a table of Type B. From these we derive 100 preliminary values of logarithms, and we write them down in form of a square. The 100 values are subtracted from the 100 corresponding 'exact' values, and the 100 differences (or 'errors') are written down in a similar manner. As some of the differences are negative, it is convenient to use either the number 9 or, still better (in accordance with the late Professor T. N. Thiele's suggestion), the letter ν as representing a negative unit prefixed to a number, the following decimal parts being positive. By adding the vertical and the horizontal arrays we

Log 6100 . . . 6199.

	(+)	(+)									
(−) 1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	
1	4	1	9	6	3	0	7	4	1	8	
2	4	1	9	6	3	0	7	4	1	8	
(−) 3	5	2	0	7	4	1	8	5	2	9	
(−) 4	6	3	1	8	5	2	9	6	3	0	
4	7	4	2	9	6	3	0	7	4	1	
5	7	4	2	9	6	3	0	7	4	1	
(−) 6	8	5	3	0	7	4	1	8	5	2	
6	9	6	4	1	8	5	2	9	6	3	
	9	6	4	1	8	5	2	9	6	3	
	298	412	ν514	ν605	ν684	ν751	ν807	ν852	ν885	ν906	ν8714
	010	123	ν224	ν313	ν391	ν457	ν512	ν555	ν587	ν608	ν5780 (+)
	722	833	ν933	021	098	163	217	259	290	309	2845
	434	544	ν643	ν730	ν805	ν869	ν922	ν963	ν992	011	ν9913
	145	254	ν352	ν438	ν512	ν575	ν626	ν666	ν695	ν712	ν6975 (+)
	ν857	ν965	ν061	ν146	ν219	ν281	ν331	ν370	ν397	ν413	ν4040 (+)
	568	675	ν770	ν854	ν926	ν986	035	073	099	114	1100
	279	385	ν479	ν561	ν632	ν692	ν739	ν776	ν801	ν815	ν8159
	ν990	095	ν188	ν269	ν339	ν397	ν444	ν479	ν503	ν516	ν5220 (+)
	701	805	ν896	ν976	045	102	148	182	205	216	2276
Σ	3004	4091	ν5060	ν5913	ν6651	ν7273	ν7781	ν8175	ν8454	ν8620	
	(−)	(−)									

7·095
ν82·015

25·080−22=3·080

2·344
ν78·801

23·543−25=ν8·543

ν97·398
ν79·901

17·497−19=ν8·497

p	V	H	*q*
ν5			5
	4091	ν4040	
ν6			4
	3004	ν5220	
ν7			3
	ν8620	ν5780	
ν8			2
	ν8454	ν6975	
ν9			1
	ν8175	ν8159	
0			0
	ν7781	ν8714	
1			ν9
	ν7273	ν9913	
2			ν8
	ν6651	1100	
3			ν7
	ν5913	2276	
4			ν6
	ν5060	2845	
5			ν5
	ν4091	4040	
6			ν4
	ν3004	5220	
			ν3
		5780	
			ν2
		6975	

can obtain the sums V and H. These sums we arrange according to magnitude in two parallel columns, the former to the left, with the 10 terms decreasing from the top downwards, the latter to the right, with the 10 terms increasing from the top downwards. (Practically we always find that the difference between two arbitrary terms of each column will be less than 10, numerically; if not, we use another preliminary set of values satisfying this condition). We further extend the two columns downward by the repetition of some of the terms at the top, having previously subtracted 10 from each of the values V and added 10 to each of the values H.

If now we wish to alter the preliminary table in order to obtain the final table, we may represent the resulting changes in the 100 values of logarithms by following a certain number (m) of the vertical arrays and a certain number (n) of the horizontal arrays, adding 1 to the values of the former and subtracting 1 from the values of the latter. In the corresponding arrays of the set of 100 errors corresponding changes will take place, subtractions in the vertical and additions in the horizontal arrays. The resulting improvement (the diminution of the sum of the squares of errors) we can easily find by the formula :

$$I=2[V]-2[H]-10m-10n+2mn,$$

[V] being the sum of the m quantities V, and [H] the sum of the n quantities H. It is now at once obvious that in order to find these quantities in the columns, we must proceed from the top of each column downwards to a certain point, without skipping over any terms. Thus the only problem left is to find the points where we are to stop, or, in other words, the numbers m and n. But if we are to stop between V_m and V_{m+1}, and between H_n and H_{n+1}, the following conditions—

$$V_m > 5-n > V_{m+1}$$
$$H_n < m-5 < H_{n+1}$$

—must be satisfied, as, otherwise, it would be better to take one step more or less, down the right or the left column. If we use the numbers

$$p=m-5$$
$$q=5-n$$

instead of m and n for the numeration of the successive intervals of the columns H and V (see example), the two conditions and also the expression of I take the simple forms :

$$V_m > q > V_{m+1}$$
$$H_n < p < H_{n+1}$$
$$I/2=[V]-[H]-(pq+25).$$

It is now easy to find out the only pairs of values of p and q compatible with the two conditions (giving what may be called the relative maxima of I). The result may be marked by limiting lines, as shown in the example. For the final choice we must, by means of the formula, calculate the corresponding values of I, ordinarily very few in number, taking the values $pq+25$ from a small table with two entries. The greatest value of I/2 is chosen. The signs (+) and (−) indicate the resulting deviations from the preliminary table.

It happens in a few cases that the number of decimal places (here 7) used in the calculation turns out to be insufficient, but it is easy to give a rule covering these cases.

If this sort of calculation is to be undertaken on a large scale it is best to try to get rid of part of the work, for example, by finding the sum of 10 function-values with equidistant arguments, without actually undertaking the addition. For special functions, such as antilogarithms, the way to proceed is obvious. Furthermore, the function considered in most cases is so nearly linear that we can find the mean of the 10 values from the value corresponding to the mean-argument, applying, if necessary, a small and easily determinable correction.

A set of 4-figure tables [1] of the type described, has been calculated by H. C. Nybølle, Mathematical Assistant at the Danish Statistical Department, and myself. Another collection also containing 5-figure tables is at present being elaborated.

I would only like to mention that similar principles might possibly find application for the construction of tables of some simple and practically important functions of complex variables.

5. Concerning the mean-square of errors in tables of Type B_1, I have tried unsuccessfully to find the exact value

[1] A. K. Erlang, *Fircifrede Logaritmetavler*, G. E. C. Gad, København. (1910-11), three editions, A, B, C, the last being the most complete.

as in the cases A and B (see above), especially for the purpose of finding the difference in this respect between B and B_1. The solution of this problem is theoretically possible, under the same supposition as to the nature of the function as above, and the integrations necessary are very simple; but the number of cases to be considered is very great. Some indications may, however, be had from the experiences available; thus we find, that the improvement I (for a set of 100 values) is, on the average, about 2 or 3 units; sometimes, although seldom, it will be as great as 50 (about), sometimes 0. We might also consider the case in which the second differences of the function are considerable (although one can, of course, get rid of this case by altering the interval). In this case the mean-square of errors produced by the aforesaid cause will obviously be about four times greater for Type B than for Type B_1.

It seems probable that Types B or B_1 will be much used in the future for the construction of tables of different functions, and if stress is laid on the greatest accuracy compatible with the arrangement, space, and number of figures chosen, Type B_1 should be preferred.

EXTENSION OF ACCURACY OF MATHEMATICAL TABLES BY IMPROVEMENT OF DIFFERENCES

W. F. SHEPPARD, Sc.D., LL.M.

1. In a paper published some years ago[1] I explained a method of extending or checking the accuracy of a mathematical table. The method is applicable (for a more or less limited range of values of x) to any function u which satisfies a linear differential equation with regard to x; it is most conveniently applied when this equation contains one derivative only.

Suppose, for example, that we wished to improve the table[2] of values of $G_0(x)$, the second Bessel function of order 0. Writing (as suggested on page 447 of the above-mentioned paper)

$$u \equiv \sqrt{x} \,.\, G_0(x),$$

we shall have

$$\frac{d^2u}{dx^2} = -\left(1 + \frac{1}{4x^2}\right)u.$$

The method then consists of the following stages :—

(i) From the ascertained values of u, calculate the values of d^2u/dx^2 by means of the above equation.

[1] 'A Method for extending the accuracy of certain mathematical tables.'—*Proceedings of the London Mathematical Society* (1899), vol. xxxi, pp. 423-448.

[2] *British Association* (1913) *Report*, pp. 115-130.

(ii) Thence deduce the 2nd (central) difference $\delta^2 u$ by the formula

$$\delta^2 u=(1+\tfrac{1}{12}\delta^2-\tfrac{1}{240}\delta^4+\tfrac{31}{60480}\delta^6-\tfrac{289}{3628800}\delta^8+\ .\ .\ .)h^2 d^2u/dx^2,$$

where h is the interval of entry of x.

(iii) Starting with the more accurate values of u_0 and $\delta u_{\frac{1}{2}}(\equiv u_1-u_0)$, obtain new values of u_2, u_3, . . . by successive additions of $\delta^2 u_1$, $\delta^2 u_2$, . . .

(iv) The process may then be repeated, if desired.

The values obtained must be checked at intervals, since the necessary error in the last figure of $\delta u_{\frac{1}{2}}$ is cumulative; and we must always use two or three more figures (at least) than will be required in the ultimate table.

2. My present object is to call attention to a special class of cases in which stage (ii) of the above process may be replaced by multiplication by a factor which, for each value of x, remains unaltered for successive improvements. The simplest case is that of e^x or e^{-x}, where successive values can be obtained by multiplication by a constant factor. For $\cosh x$, $\sin x$, etc., 2nd differences must be used. For example,

$$\delta^2 \sin x=-2(1-\cos h)\ .\ \sin x,$$

so that when $\sin x$ is tabulated the 2nd differences are obtained, to a larger number of decimal places, by multiplication by a constant factor. *Example* 1 illustrates the application of the method to extending a 5-figure table of $\sin x$, arranged by intervals of $\cdot 01\times\frac{1}{2}\pi$, to (approximately) 11 places. Col. (0) gives the values of $x/\frac{1}{2}\pi$, from $\cdot 50$ to $\cdot 60$; and col. (1) gives the 5-figure values of $u\equiv\sin x$. The value of $\delta^2 u$ is to be obtained from that of u by multiplying it by $-2(1-\cos h)=-\cdot 00024\ 67350\ 36679$. The values to 6 significant figures (the 6th being only approximate) are given in col. (2). Cols. (3) and (4) show the deduced values of δu

and u, starting with more accurate initial values; the values to 13 places being, according to Callet,

$$(x=\cdot 500)\quad u=\cdot 70710\,67811\,865,$$
$$(x=\cdot 505)\quad \delta u=\cdot 01101\,95165\,766.$$

From col. (4) we obtain, in col. (5), better values of $\delta^2 u$, and then repeat the process, obtaining in col. (7) values of u which will be approximately correct to 11 places. The value obtained for $x=\cdot 60\times\frac{1}{2}\pi$ is $\cdot 80901\,69943\,889$, the true value being $\cdot 80901\,69943\,749$.

3. The method is applicable to any u which satisfies an equation

$$\frac{1}{u}\frac{du}{dx}=f(x),$$

where $f(x)$ and its derivatives can be calculated for any value of x; for this equation gives

$$\frac{1}{u}\frac{d^2u}{dx^2}=f'(x)+\{f(x)\}^2,$$

$$\frac{1}{u}\frac{d^3u}{dx^3}=f''(x)+3f(x)f'(x)+\{f(x)\}^3, \text{ etc.}$$

so that the ratio of

$$\delta^2 u\equiv h^2u''+\tfrac{1}{12}h^4u^{\text{iv}}+\tfrac{1}{360}h^6u^{\text{vi}}+\tfrac{1}{20160}h^8u^{\text{viii}}+\ \dots$$

to u can be calculated for each value of x.

4. Similarly, if u is the integral of a function satisfying such an equation, we can obtain the ratio of $\delta^3 u$ to δu for each value of x (intermediate between the entered values), and thus improve the table of u by improving the 3rd differences.

As an example, let us take

$$z \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad u \equiv \int_{-\infty}^{x} z\,dx.$$

This gives

$$\delta u = z\{h + \tfrac{1}{24}(x^2-1)h^3 + \tfrac{1}{1920}(x^4-6x^2+3)h^5 + \tfrac{1}{322560}(x^6-15x^4+45x^2-15)h^7 + \ldots\},$$

$$\delta^3 u = z\{(x^2-1)h^3 + \tfrac{1}{8}(x^4-6x^2+3)h^5 + \tfrac{13}{1920}(x^6-15x^4+45x^2-15)h^7 + \tfrac{41}{193536}(x^8-28x^6+210x^4-420x^2+105)h^9 + \ldots\},$$

whence

$$\delta^3 u/\delta u = (x^2-1)h^2 + \tfrac{1}{12}(x^4-8x^2+4)h^4 + \tfrac{1}{360}(x^6-24x^4+93x^2-31)h^6 + \tfrac{1}{60480}(3x^8-144x^6+1716x^4-4376x^2+1094)h^8 + \ldots$$

Example 2 gives the application of the method to improving the values of u from $x=1{\cdot}60$ to $x=1{\cdot}80$, for $h={\cdot}01$. Col. (1) gives the ratio of $\delta^3 u$ to δu, to 9 significant figures; the third term in the above formula only affects the 9th figure. The values of u in col. (2) are taken from my table in *Biometrika*, vol. ii, p. 184; and the values of δu in col. (3) are the differences of these. The process of obtaining col. (9) from col. (3) is similar to that described in § 4 above, the starting values being

$$(x=1{\cdot}605)\ \delta u = +{\cdot}00110\ 03635\ 51438,$$
$$(x=1{\cdot}60\ \ )\ \delta^2 u = -{\cdot}00001\ 77472\ 68473.$$

The values of u are then obtained in col. (10), the starting value being (I am not positive as to the final figure)

$$(x=1{\cdot}60)\ u = {\cdot}94520\ 07083\ 004.$$

In col. (9) δu is obtained to 15 places, but the last two are only rough, and therefore in constructing col. (10) the values in col. (9) are only taken to 13 places. The value of u obtained for $x=1{\cdot}80$ is ·96406 96808 880, the true value to 13 places being ·96406 96808 871.

Example 1.—Improvement of sin x.

$u=\sin x$; $h=\cdot 01\times\frac{1}{2}\pi$; $d^2u/u=-\cdot 00024\ 67350\ 36679$.

The dots in cols. (5), (6), and (7) represent the figures in ordinary type in cols. (2), (3), and (4) respectively.

$x/\frac{1}{2}\pi$	u (original value)	$10^9\delta^2u$	$10^9\delta u$	u	$10^{13}\delta^2u$	$10^{13}\delta u$	u
(0)	(1)	(2) −	(3) +	(4)	(5) −	(6) +	(7)
·50	·*70711*	17 *4469*		·70710 *6781*	. . 44680 175		. . 67811 865
			1101 *9517*			. . 95165 766	
1	·*71813*	17 *7188*		·71812 *6298*	. . 71869 185		. . 62977 631
			1084 *2329*			. . 23296 581	
2	·*72897*	17 *9862*		·72896 *8627*	. . 98621 009		. . 86274 212
			1066 *2467*			. . 24675 572	
3	·*73963*	18 *2493*		·73963 *1094*	. . 24929 051		. . 10949 784
			1047 *9974*			. . 99746 521	
4	·*75011*	18 *5078*		·75011 *1068*	. . 50786 819		. . 10696 305
			1029 *4896*			. . 48959 702	
5	·*76041*	18 *7620*		·76040 *5964*	. . 76187 934		. . 59656 007
			1010 *7276*			. . 72771 768	
6	·*77051*	19 *0112*		·77051 *3240*	. . 01126 125		. . 32427 775
			991 *7164*			. . 71645 643	
7	·*78043*	19 *2559*		·78043 *0404*	. . 25595 244		. . 04073 418
			972 *4605*			. . 46050 399	
8	·*79016*	19 *4960*		·79015 *5009*	. . 49589 251		. . 50123 817
			952 *9645*			. . 96461 148	
9	·*79968*	19 *7309*		·79968 *4654*	. . 73102 224		. . 46584 965
			933 *2336*			. . 23358 924	
·60	·*80902*	19 *9614*		·80901 *6990*	. . 96128 367		. . 69943 889

EXAMPLE 2. IMPROVEMENT OF

x	$10^{12}\times\frac{\delta^3 u}{\delta u}$	u (original value)	$10^7\delta u$	$10^{11}\delta^3 u$	$10^{11}\delta^2 u$	$10^{11}\delta u$
(0)	(1)	(2)	(3)	(4)	(5)	(6)
			+	+	−	+
1·60		·9452007			1 77472 7	
	157594190		11004	17342		110 03635 5
1		·9463011			1 75738 5	
	160814114		10828	17413		108 27897 0
2		·9473839			1 73997 2	
	164054040		10654	17478		106 53899 8
3		·9484493			1 72249 4	
	167313967		10481	17536		104 81650 4
4		·9494974			1 70495 8	
	170593895		10311	17590		103 11154 6
5		·9505285			1 68736 8	
	173893825		10143	17638		101 42417 8
6		·9515428			1 66973 0	
	177213756		9975	17677		99 75444 8
7		·9525403			1 65205 3	
	180553689		9810	17712		98 10239 5
8		·9535213			1 63434 1	
	183913623		9647	17742		96 46805 4
9		·9544860			1 61659 9	
	187293559		9485	17765		94 85145 5
1·70		·9554345			1 59883 4	
	190693496		9326	17784		93 25262 1
1		·9563671			1 58105 0	
	194113434		9167	17794		91 67157 1
2		·9572838			1 56325 6	
	197553375		9011	17802		90 10831 5
3		·9581849			1 54545 4	
	201013317		8856	17802		88 56286 1
4		·9590705			1 52765 2	
	204493260		8703	17797		87 03520 9
5		·9599408			1 50985 5	
	207993205		8553	17790		85 52535 4
6		·9607961			1 49206 5	
	211513153		8403	17773		84 03328 9
7		·9616364			1 47429 2	
	215053101		8256	17755		82 55899 7
8		·9624620			1 45653 7	
	218613052		8110	17730		81 10246 0
9		·9632730			1 43880 7	
	222193005		7967	17702		79 66365 3
1·80		·9640697			1 42110 5	

$$u \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx.$$

$10^{15}\delta^3 u$	$10^{15}\delta^2 u$	$10^{15}\delta u$	u
(7) +	(8) −	(9) +	(10)
	1 77472 68473		·94520 07083 004
1734 10902		110 03635 51438	
	1 75738 57571		·94630 10718 518
1741 27866		108 27896 93867	
	1 73997 29705		·94738 38615 457
1747 81530		106 53899 64162	
	1 72249 48175		·94844 92515 099
1753 72651		104 81650 15987	
	1 70495 75524		·94949 74165 259
1759 02003		103 11154 40463	
	1 68736 73521		·95052 85319 664
1763 70383		101 42417 66942	
	1 66973 03138		·95154 27737 333
1767 78604		99 75444 63804	
	1 65205 24534		·95254 03181 971
1771 27493		98 10239 39270	
	1 63433 97041		·95352 13421 364
1774 17893		96 46805 42229	
	1 61659 79148		·95448 60226 786
1776 50666		94 85145 63081	
	1 59883 28482		·95543 45372 417
1778 26683		93 25262 34599	
	1 58105 01799		·95636 70634 763
1779 46834		91 67157 32800	
	1 56325 54965		·95728 37792 091
1780 12017		90 10831 77835	
	1 54545 42948		·95818 48623 869
1780 23145		88 56286 34887	
	1 52765 19803		·95907 04910 218
1779 81136		87 03521 15084	
	1 50985 38667		·95994 08431 369
1778 86925		85 52535 76417	
	1 49206 51742		·96079 60967 133
1777 41459		84 03329 24675	
	1 47429 10283		·96163 64296 380
1775 45683		82 55900 14392	
	1 45653 64600		·96246 20196 524
1773 00563		81 10246 49792	
	1 43880 64037		·96327 30443 022
1770 07064		79 66365 85755	
	1 42110 56973		·96406 96808 880

UNPUBLISHED TABLES RELATING TO THE PROBABILITY-INTEGRAL

W. F. SHEPPARD, Sc.D., LL.M.

The earlier tables of the probability-integral were based upon

$$z=\frac{1}{\sqrt{\pi}}e^{-x^2}$$

as the equation to the curve of error. A list of these tables, of which that published by J. Burgess in 1898[1] is the most complete, is given by J. W. L. Glaisher in his article on 'Table, mathematical,' in the *Encyclopædia Britannica*, 11th ed., xxvi, at p. 335.

In order to use these tables for statistical and other purposes, deviations from the mean have to be expressed in terms of the 'modulus,' which is $\sqrt{2}$ times the 'standard deviation' (square root of mean square of deviation). Statisticians usually find it more convenient to use the standard deviation, which implies comparison with a figure whose bounding curve has the equation—obviously a more natural one algebraically—

$$z=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}.$$

Tables relating to this figure are therefore of practical importance.

The following is a list of tables which I have constructed on this basis. Some of them have been published in *Biometrika*; I include these here, as they are not mentioned either

[1] *Trans. Roy. Soc. Edin.*, vol. xxxix.

in Glaisher's article or in the Royal Society's Index. The notation is

$$z \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad a \equiv 2\int_0^x z\,dx, \quad u \equiv \log_{10}\frac{1+a}{1-a},$$

so that the ordinate z divides the figure into two portions whose areas are $\frac{1}{2}(1+a)$ and $\frac{1}{2}(1-a)$, and u is the difference of the logarithms (to base 10) of these areas.

The number in col. (6) of the list is the number of decimal places to which values have been obtained accurately, by an independent check of the final figure in doubtful cases.

LIST OF TABLES

Argument.	Values of argument.			Entry.	Number of decimal places in entry.	
	First.	Interval.	Last.		Accurate.	Approximate.
(1)	(2)	(3)	(4)	(5)	(6)	(7)
x	·000	·005	2·550	z	..	9 to 10
,,	·00	·01	4·50	,,	7[1]	9 to 11
,,	4·50	·01	6·00	,,	10[1]	12 to 14
x	·000	·001	1·000	$\frac{1}{2}(1+a)$	..	10 to 11
,,	1·000	·005	2·550	,,	..	11 to 12
,,	·00	·01	4·50	,,	7[1]	10 to 12
,,	4·50	·01	6·00	,,	10[1]	10 to 13
x	·00	·01	6·00	$\log_{10}\frac{1}{2}(1-a)$	7	10
a	·000	·002	·998	x	4[2]	7
,,	·00	·01	·80	,,	7[1]	11
a	·000	·005	·830	z	..	9
,,	·00	·01	·80	,,	7[1]	..
u	·00	·01	6·00	x	7	9
u	·00	·01	5·00	z	..	7 to 9

[1] *Biometrika* (1903), vol. ii, pp. 174-190. [2] *Ibid.* (1907), vol. v, pp. 404-406.

A METHOD OF FINDING WITHOUT THE USE OF TABLES THE NUMBER CORRESPONDING TO A GIVEN NATURAL LOGARITHM

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The writer had read at the Third Summer Meeting of the American Mathematical Society, held at Buffalo, N.Y., in 1896, a paper entitled 'A Method of Finding, without Tables, the Number corresponding to a Given Logarithm,' in which only common logarithms were considered. It is the purpose of the present note to show how to find in a similar way the number corresponding to a given *natural* logarithm, base e.

Let a be the given logarithm, and N the number required, so that $\log_e N = a$.

(1) Assume $$N = e^m(1+x) = e^{\log N} = e^a,$$

then $$1+x = e^{a-m} = 1+(a-m)+\frac{(a-m)^2}{2}+ \text{etc.}$$

and $$N = e^m\left\{1+(a-m)+\frac{(a-m)^2}{2}+\ \ldots\ \right\}.$$

(2) With $N = e^{m+1}(1-y) = e^a$

we get by similar expansion

$$N = e^{m+1}\left\{1+(m+1-a)+\frac{(m+1-a)^2}{2}+\ \ldots\ \right\}.$$

If an integer m can be assigned so that $a-m$ or $m+1-a$ is small the series becomes rapidly convergent.

Thus, to find the number whose natural logarithm is 2·0014805, take $m=2$. Then

$$N=e^2[1+\cdot0014805+\tfrac{1}{2}(\cdot0014805)^2+\tfrac{1}{6}(\cdot0014805)^3+ \text{ to the 5th power}]$$

$$=7\cdot4000035+.$$

(3) To obtain more general expressions, put

$$N=e^m(1+u)^n=e^a$$

$$1+u=e^{(a-m)/n}$$

$$\text{and } N=e^m\left[1+\frac{a-m}{n}+\frac{1}{2}\left(\frac{a-m}{n}\right)^2+\ldots\right]^n$$

where n may have any convenient integral value.

(4) Similarly if $N=e^{m+1}(1-t)^n=e^a$

we find

$$N=e^{m+1}\left\{1-\frac{m+1-a}{n}+\frac{1}{2}\left(\frac{m+1-a}{n}\right)^2-\ldots\right\}^n.$$

The m series (1) and (3) are to be used when a is nearer to m than to $(m+1)$; and the $(m+1)$ series (2) and (4) are to be used when a is nearer to $(m+1)$ than to m.

APPROXIMATE DETERMINATION OF THE FUNCTIONS OF AN ANGLE, AND THE CONVERSE

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Of the art and science of engineering there is no branch of mathematics, except the rudiments, that receives so varied and constant application as does that of trigonometry, the basis of which is a series of tables showing the relative numerical values of the functions of equi-distant angles of the quadrant of a circle.

The perfecting of these tables more or less engaged the attention of mathematicians for several centuries, and it was not until the labour of George Joachim Rheticus was completed, and forty years later published (1613), that they attained their highest degree of efficiency. The magnitude and importance of his latest work, in which the sines and cosines of every tenth second of the quadrant are calculated to the fifteenth decimal, can be the better appreciated when it may be said of it that it is probably the only scientific work that has maintained its pre-eminence since the date of its publication to the present time.

Without these tables in one form or another the scientist and the engineer are practically helpless. It is safe to say that the occasion has arisen in the experience of every practical engineer when he would have welcomed a substitute for the tables, however empirical it might be.

Of the several functions of an angle the sines and cosines may be said to be independent, the others being directly or indirectly derived from them. Not for this reason alone is our attention directed to the sines and cosines, but also because their range of values is so much smaller than that of the tangents, secants, etc. Now in the sines and cosines, and much more so in the other functions, a comparison of their numerical relations will show a range of values of so great a variation as to preclude any simple order of progression approximating to them. If, however, we divide the sines by their respective angles we have a succession of numbers, the greatest of which is not quite 1·6 times the smallest. These numbers may be called the Initial Factors.

A table giving the sines and initial factors of the angles 1°, 10°, 20°, and so on to 90°, may easily be constructed, when it will be seen that the initial factors range from ·017453 to ·011111. By consideration of the first and second differences the following formula is found to be applicable to the calculation of the sines of angles up to 45°, with an error of not more than 3 in 5000, usually less:

$$\text{Sin } A^\circ = A(\cdot 01745 - \cdot 00000086A^2).$$

Another formula giving values over the whole quadrant to an accuracy sufficient for most practical purposes is

$$\text{Sin } A^\circ = A\left(\cdot 01745 - \frac{A^2 \times 10^{-5}}{11 + \cdot 01A + \cdot 0001A^2}\right).$$

More accurate results may be obtained by means of the modified form

$$\text{Sin } A^\circ = A\left(\cdot 0174533 - \frac{A^2 \times 10^{-5}}{11\cdot 27 + 11A^2/60000}\right).$$

An obvious extension of the method is to apply the same principle to limited ranges, assigning to each range appropriate constants, which may be tabulated for use on a single page.

The converse problem is to derive the angle from the given function. Whatever be the particular function whose value is given, it is a mere matter of arithmetic to calculate the sine and cosine. When this is effected the angle may be calculated with an accuracy to within an error of half a minute by the following formulæ, the first of which is applicable to angles less than 45°, and the second to angles greater than 45°—

$$A^\circ = \frac{\text{Sine } A}{\cdot 01111 + \cdot 00634 \text{ cosine } A} \qquad \text{A less than } 45^\circ$$

$$A^\circ = \frac{\text{Sine } A}{\cdot 01147 + \cdot 006 \text{ cosine } A} \qquad \text{A greater than } 45^\circ.$$

For the application of more accurate formulæ along the same lines the author has prepared tables of entry for every half-degree, which can be used as a substitute for the great tables of Rheticus in obtaining, with the same accuracy, the sines of all angles within the quadrant.


A H

LIFE PROBABILITIES: ON A LOGARITHMIC CRITERION OF DR GOLDZIHER, AND ON ITS EXTENSION

ALBERT QUIQUET, General Secretary of the
Institute of French Actuaries

I

Actuaries could not remain indifferent to the commemoration of the invention of logarithms.[1] The celebration of its tercentenary, so successfully organised in Edinburgh, has, thanks to the kindness of the Committee, left a place for actuarial science amid the works destined to honour JOHN NAPIER.

I do not intend to prove here what an important place logarithms hold in the actuarial professions. Here, as elsewhere, they are one of the most powerful means of numerical calculation. This short article will be limited to certain very special applications of logarithms in relation to life probabilities.

II

In 1905, a young Hungarian actuary—Dr Charles Goldziher — made some very sound remarks,[2] which deserve to be better known, on the use of Gompertz's law, as extended by Makeham. We know that this famous law

[1] *Logarithmorum Canonis Mirifici Descriptio*, 1614.

[2] Dr Goldziher Károly: *A Halandósági Tábla Kiegyenlitése* (*L'ajustement des Tables de mortalité*), Budapest, 1905. See also les *Comptes Rendus hebdomadaires des Séances de l'Académie des Sciences de Paris*, séance du 30 Octobre, 1905.

is widely used to represent, by an algebraic formula, the number of survivors at each age, as shown in a mortality table. When direct observations have yielded a series of unadjusted rates of mortality, they present irregularities which render them difficult to handle as they stand. We endeavour therefore to *adjust* them, and several methods are available. But if we confine ourselves to Makeham's law for the adjustment, how are we to know whether this choice is justifiable? Dr Goldziher has attempted to find a criterion for this.

In practice, the constants of the law are determined by means of 'age groups,' and we compare the systems of results deduced from different groups. The next point is to discuss whether these results agree sufficiently amongst themselves, or, rather, whether their differences are of the same order as the errors which are inevitable in the unadjusted observations. Any process which will facilitate the comparison of these results will afford a real relief to a laborious task. Dr Goldziher has indicated to us two of them.

At the age of x, according to Makeham's law, the probability of surviving for a year is expressed by the formula

$$p_x = sg^{c^x(c-1)}$$

in which s, g, c are three constants. Dr Goldziher has endeavoured to find a criterion that can be applied, not directly to the values of p_x, but to certain functions of p_x, which can be expressed as *linear* functions of x. This permits of a very simple mode of representation by a certain straight line; and it will then only remain to ascertain whether the *observations*, that is to say, the unadjusted values of the function of p under consideration, range themselves graphically in accordance with such a straight line, or only differ therefrom to a negligible extent.

Here are the two distinct formulæ which Dr Goldziher has chosen to obtain a criterion of this kind.

Taking as ordinate the following function of the *observations* p_x, namely,

(1) $$y=\log(\log p_x-\log p_{x+t}),$$

we arrive at the criterion

$$y=x\log c+\log(c^t-1)+\log m.$$

Alternatively taking as ordinate the function

(2) $$y=\log(\log s-\log p_x),$$

we arrive at the criterion

$$y=x\log c+\log m.$$

In both these expressions

$$m=(c-1)(-\log g)$$

which is constant.

The symbol 't,' in the first criterion, designates the interval which has been agreed upon beforehand in order to space out the different 'age groups' combined with a view to determining the constants. For example t may be taken as

$$t=4, \text{ or } t=5$$

and once 't' is fixed, the first criterion contains only two variables, namely x and y.

One method given by Dr Altenburger[1] allows us besides to reduce the interval t to $t=1$.

With regard to the second criterion, the constant 's' must

[1] Julius Altenburger: *Beiträge zum Problem der Ausgleichung von Sterblichkeitstafeln.* (Extrait des *Mitteilungen des Österreichisch-ungarischen Verbandes der Privat-Versicherungs-Anstalten,* Vienne, 1905.) In this important work will be found applications of both criteria.

be determined by a previous investigation, of which Dr Goldziher has given an outline.

III

Previous to Dr Goldziher, various criteria had already been suggested. Gompertz and Makeham themselves had indicated some of them. Mr Ryan mentioned this in 1892 in an obituary article on Makeham.[1]

If we designate by ${}_hp_x$ the probability at the age of x of living, not one year, but h years, and if we find by observation the values of

$$ {}_{20}p_{20} \quad {}_{20}p_{40} \quad {}_{20}p_{60}, $$

the logarithms of these three quantities, according to Gompertz's law, should be in geometric progression. This is one criterion.

Denoting these logarithms by a, b, c, respectively, Makeham has shown that by adding to each of them a quantity ρ equal to $\frac{b^2-ac}{a+c-2b}$ we shall get a geometric progression,

$$ a+\rho \quad b+\rho \quad c+\rho. $$

The constancy of ρ is also a criterion.

IV

In the thesis which I submitted to the Institute of French Actuaries in order to become a Fellow[2] I devoted the first nine pages of Chapter II to 'criteria.' These enable us to ascertain whether a mortality table can be

[1] *The late William Mathew Makeham.* (*Journal of the Institute of Actuaries*, xxx, April, 1892.

[2] Albert Quiquet: *Représentation algébrique des Tables de Survie; généralisation des lois de Gompertz, Makeham,* etc., Paris, 1893. See also by same author: *Aperçu historique sur les formules d'interpolation des Tables de Survie et de Mortalité,* Paris, 1893.

represented by one of the functions (generalising those of Gompertz and of Makeham) which I have called 'Mortality Functions of the order n.'

Denoting by z_x the first difference of $\log p_x$ I proved that the numerical values of the z's had to satisfy a certain equation with constant coefficients:

$$B_0 z_x + B_1 z_{x+1} + \ldots\ldots + B_n z_{x+n} = 0 \tag{3}$$

This is again a criterion.[1]

I also showed how this criterion could be made independent of $B_0, B_1, \ldots\ldots B_n$. It was sufficient to consider the system:

$$\begin{aligned} B_0 z_x + B_1 z_{x+1} + \ldots\ldots + B_n z_{x+n} &= 0 \\ B_0 z_{x+1} + B_1 z_{x+2} + \ldots\ldots + B_n z_{x+n+1} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ B_0 z_{x+n} + B_1 z_{x+n+1} + \ldots\ldots + B_n z_{x+2n} &= 0 \end{aligned} \tag{4}$$

to deduce from same:

$$\begin{vmatrix} z_x & z_{x+1} & \ldots\ldots & z_{x+n} \\ z_{x+1} & z_{x+2} & \ldots\ldots & z_{x+n+1} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ z_{x+n} & z_{x+n+1} & \ldots\ldots & z_{x+2n} \end{vmatrix} = 0 \tag{5}$$

another form of criterion where only the *observed* values appear.

[1] A particular instance of this criterion is that in which the scale of recurrence (3) of the functions z_x has only one multiple root of the order n and is equal to unity. Z_x is then a simple polynomial of the degrees $n-1$. I had occasion to quote this polynomial at the fifth International Congress of Mathematicians, held at Cambridge in 1912. In a communication entitled: *Sur une Méthode d'interpolation exposée par Henri Poincaré et sur une application possible aux fonctions de survie d'ordre n*, I summarised a lecture of Henri Poincaré, reproducing as nearly as possible the demonstration which he had given, and wishing above all to render homage to the memory of the master who had just passed away from us. The method itself we owe to Tchebycheff: M. de Savitch and M. Achard were the first to bring it to my notice.

V

To return to the line of thought which guided Dr Goldziher, principally in his first criterion, it would be well to apply criterion (5) to ages taken at intervals of t (years). This extension is simple. In place of system (4) where we pass from each equation to the next by adding 1 to x, let us consider system (6) in which we pass from each equation to the next by adding t to x.

$$(6) \qquad \begin{array}{l} B_0 z_x + B_1 z_{x+1} + \dots\dots + B_n z_{z+n} = 0 \\ B_0 z_{x+t} + B_1 z_{x+t+1} + \dots\dots + B_n z_{x+t+n} = 0 \\ \dots\dots\dots\dots\dots\dots \\ B_0 z_{x+nt} + B_1 z_{x+nt+1} + \dots\dots + B_n z_{x+n(t+1)} = 0 \end{array}$$

the criterion becomes :

$$(7) \qquad \begin{vmatrix} z_x & z_{x+1} & \dots\dots & z_{x+n} \\ z_{x+t} & z_{x+t+1} & \dots\dots & z_{x+t+n} \\ \dots & \dots & \dots & \dots \\ z_{x+nt} & z_{x+nt+1} & \dots\dots & z_{x+n(t+1)} \end{vmatrix} = 0$$

In this form we notice immediately that $z_x, z_{x+t}, \dots$, z_{x+nt}, are connected, whatever x may be, by a single linear and homogeneous relation.

To sum up, it is advisable to form equations, such as (7), in sufficiently large numbers for all the numerical values destined to be included in the interpolation formula to occur at least once.[1] This series of equations (7) lays down the necessary and sufficient conditions that the observations are consistent with a mortality table formed from a mortality function of the order n.

When it is a question of actual application, these equations will never be strictly satisfied. When *observed* values are inserted the determinants of the first member will not,

[1] For the special value $t=n+1$, all the indices differ in (7), and consequently none of the z's are repeated in the same determinant.

as a rule, be exactly equal to zero. Such a result would only occur if the observations were in absolute accordance with an algebraic function, and no empirical table agrees so strictly as this with mathematical analysis. The determinants will therefore be, almost certainly, different from zero. But, if they differ little, or better still, if they oscillate about zero, this occurrence, with the usual reservations, would warrant the admission that equations such as (7) are *sufficiently* complied with.

To keep within practicable limits one is obliged to stop at the first values of n,

$$n=1,\ n=2,\ .\ .\ .\ .\ .,$$

already confirmed otherwise by experimental examination. The hypothesis $n=1$ leads to the laws of Gompertz and Makeham. The hypothesis $n=2$ yields four laws of mortality which I have examined in some detail in the aforementioned thesis.

The z's which have just been mentioned, as well as the y's which define the functions (1) and (2) of Dr Goldziher involve an extensive use of logarithms in the numerical work. This is why I have considered it suitable to point them out to-day in recognition of our gratitude to the memory of Napier.

AN ACCOUNT OF THE NAPIER TERCENTENARY CELEBRATION AND CONGRESS

By the Secretary, C. G. Knott, D.Sc.

At its Meeting of July 15, 1912, the Council of the Royal Society of Edinburgh resolved to commemorate the publication in 1614 of Napier's *Mirifici Logarithmorum Canonis Descriptio*, by holding a Tercentenary Celebration in Edinburgh some time in the summer of 1914.

A committee was appointed with the General Secretary, Dr C. G. Knott, as Convener, to invite other Societies, Institutions, Corporations and the like, to nominate delegates so as to constitute a representative General Committee.

A very cordial response was received from nearly all the Institutions approached; and the General Committee as finally constituted was as follows, the different representative bodies being in the order of date of acceptance :—

GENERAL COMMITTEE

Hon. Secretary

Dr C. G. Knott, Gen. Sec., Royal Society of Edinburgh, 22 George Street, Edinburgh.

Hon. Treasurer

Mr Adam Tait, Royal Bank of Scotland, Edinburgh.

Royal Society of Edinburgh

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W. Allan Carter, M.INST.C.E., F.R.S.E.
C. G. Knott, D.SC., F.R.S.E.
D. C. M'Intosh, M.A., D.SC., F.R.S.E.
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Professor CHARLES NIVEN, M.A., F.R.S.

Glasgow University

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Royal Technical College, Glasgow

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Professor JOHN MILLER, M.A., D.SC., F.R.S.E.

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George Heriot's Trust, Edinburgh

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WM. HUTTON, F.F.A.
W. R. MACDONALD, F.F.A.
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A. E. SPRAGUE, M.A., D.SC., F.F.A., F.I.A.

Merchiston Castle School, Edinburgh

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GEORGE SMITH, M.A. (OXON.)

Edinburgh Mathematical Society

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P. COMRIE, M.A., B.SC., F.R.S.E.

Royal Astronomical Society

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Professor R. A. SAMPSON, M.A., D.SC., F.R.S., F.R.S.E.

St. Andrews University

Professor SCOTT LANG, M.A., B.SC., F.R.S.E.
D. M. Y. SOMMERVILLE, M.A., D.SC., F.R.S.E.

University College, Dundee

Professor A. H. GIBSON, D.SC., F.R.S.E.
Professor J. E. A. STEGGALL, M.A., F.R.S.E.

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Major PERCY MACMAHON, D.SC., F.R.S.

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Judge MACFARLANE, J.P.

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Captain A. E. Bell. | Captain A. S. Thomson.

Institute of Accountants and Actuaries in Glasgow

Robert Carswell, writer. | D. Norman Sloan, c.a.

Solicitors Supreme Courts, Edinburgh

James Macdonald, s.s.c. | John Mathison, s.s.c.

H.M. Nautical Almanac Office

W. F. Doak, m.a., f.r.a.s.

Faculty of Procurators, Glasgow

George H. Robb.

Society of Writers to H.M. Signet

Ian M'Intyre, ll.b., w.s. | Sir Colin George Macrae, w.s.

Society of Accountants in Edinburgh

William Home Cook, c.a. | Robert C. Millar, c.a.

Institution of Mechanical Engineers

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The Institute of Actuaries

The President. | G. J. Lidstone, f.a.s.

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Dowager Lady Clark.
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A. S. Napier, Esq., Ettrick.
Colonel H. E. Napier, Chester.
Miss Napier, Chelsea.
Miss Catherine Forrester, Stirling.
George Heath, Esq., Surrey.
C. E. Heath, Esq., Surrey.
Dr and Mrs Howard Marsh, Cambridge.
Rev. Mark Napier Rice, North Cave, E. Yorkshire.
Miss Marcia Rice, Rugeley.

INDIVIDUAL MEMBERS

The Rt. Hon. THE LORD PROVOST OF EDINBURGH.
JAMES C. DUNLOP, M.D.
Sir F. W. DYSON, M.A., LL.D., F.R.S., F.R.S.E.
Professor H. H. TURNER. M.A., D.SC., D.C.L., F.R.S.

The General Committee, called together by the Council of the Royal Society of Edinburgh to consider the proposal to celebrate the Tercentenary of the publication of Napier's *Logarithms*, held their first Meeting in the Royal Society House, 22 George Street, on Saturday the 22nd of February 1913, at 11 A.M.

On the motion of Dr Knott, General Secretary of the Royal Society of Edinburgh, the Chair was taken by Mr J. R. Findlay, one of the Representatives of the Edinburgh Merchant Company.

The following Representatives were present :—

Royal Society of Edinburgh :—Dr W. B. BLAIKIE, Dr G. A. CARSE, Mr W. ALLAN CARTER, Dr C. G. KNOTT, Professor J. G. MACGREGOR, Dr J. S. MACKAY, Dr J. R. MILNE, Professor E. T. WHITTAKER.

Edinburgh Merchant Company :—Mr J. R. FINDLAY, Sir DAVID PAULIN.

Aberdeen University :—Professor H. M. MACDONALD.

Glasgow University :—Professor G. A. GIBSON, Professor A. GRAY.

Royal Technical College, Glasgow :—Dr JAMES MACKENZIE, Professor JOHN MILLER.

Faculty of Actuaries :—Mr GORDON DOUGLAS, Mr A. HEWAT, Mr W. R. MACDONALD, Mr J. J. M'LAUCHLAN, Dr A. E. SPRAGUE.

Merchiston Castle School :—Sir THOMAS CLOUSTON (for Mr B. HALL BLYTH), Mr GEORGE SMITH.

Edinburgh Mathematical Society :—Mr A. G. BURGESS, Mr P. COMRIE.

Royal Astronomical Society :—Professor R. A. SAMPSON.

St. Andrews University :—Professor SCOTT LANG.

University College, Dundee :—Professor J. E. A. STEGGALL.

Institute of Bankers :—Mr ADAM TAIT.

Edinburgh Provincial Committee for the Training of Teachers :— Mr JOHN ALISON, Dr D. F. LOWE.

Royal Society, London :—(Professor H. M. MACDONALD), Major PERCY MACMAHON.

Letters were read from Dr James Burgess, of the Royal Society of Edinburgh, and Dr J. W. L. Glaisher, of the Royal Astronomical Society,—both in hearty sympathy with the proposal.

After a general discussion as to the nature and date of the proposed Celebration, an Executive Committee was appointed, with power to add to their number, the Honorary Secretaries and Treasurer being the same as for the General Committee.

Members of the Executive Committee :—

> The Lord Provost of Edinburgh, Mr Alison, Mr Burgess, Mr Allan Carter, Mr Gordon Douglas, Mr J. R. Findlay, Professor Gibson, Mr Hewat, Professor Scott Lang, Mr Macdonald, Professor MacGregor, Dr Mackay, Professor Sampson, Mr Smith, Professor Whittaker; Drs C. G. Knott and A. E. Sprague, *Honorary Secretaries*; Mr Adam Tait, *Honorary Treasurer*.

Dr Sprague subsequently asked to be relieved of his official position as one of the Honorary Secretaries.

The death of Professor MacGregor on the 21st of May 1913, and of Dr Mackay on the 26th of March 1914, deprived the Committee of two valuable members.

The Committee was given power to carry out, as far as possible, the following suggestions of the General Committee :—

1. That a Congress be held over a week-end in June 1914, to be opened by a Public Reception and an Address on some aspect of Napier's Life and Work by an eminent man ;
2. That, in response to an invitation from the Directors of Merchiston Castle School, a Garden Party be held in the Grounds of Merchiston Castle ;
3. That Exhibits be made of all kinds of calculating machines and of logarithmic and other mathematical books which are necessary for calculation, and of objects of historic interest in connection with Napier ;

4. That a Memorial Volume be published containing the more important of the Addresses and Communications delivered at the Congress, and also possibly some of Dr Sang's manuscript Tables of logarithms and sines;
5. That eminent Mathematicians be invited from foreign countries to take part in the Celebration;
6. That, to meet preliminary expenses, a Donation List be opened, to which Societies and individuals may contribute;
7. That those interested in the proposal be asked to become Founder Members, the subscription being £2;
8. That the Ordinary Membership subscription be kept as low as possible;
9. That an effort be made to obtain a good portrait of John Napier, the sale of which would cover the expense of reproduction from an authentic painting.

The appointment of Sub-Committees to take charge of the special Departments of work indicated above was left in the hands of the Executive Committee, in whose hands also lay the convening of the General Committee, when this was thought to be necessary.

A second Meeting of the General Committee was held in the Royal Society House on Friday the 10th of October 1913, when the recommendation of the Executive Committee to hold the Tercentenary from the 24th to the 27th of July was agreed to.

It was announced that Lord Moulton had agreed to give the inaugural address.

On the motion of Mr Hewat, it was resolved to hold a Commemoration Service on Sunday the 26th of July; but it was left to the Executive Committee to decide where the Service was to be held, whether in St. Giles Cathedral or in St. Cuthbert's Parish Church.

The probable cost of the Memorial Volume having been considered, it was agreed to aim at the collection of at least

£200 by subscriptions and donations, and the Secretary was authorised to ask for donations from Societies.

A third Meeting of the General Committee was held in the Royal Society House on the 20th of May 1914. The Secretary announced that close on £500 had already been collected, and reported that in view of this hearty response, the Executive Committee recommended that a *Handbook* of the Exhibition of Napier Relics, Books, Instruments, etc., be published, to be presented to each Member of the Napier Tercentenary Celebration. This was agreed to.

The following Sub-Committees were appointed to carry out the details of the Congress, and the publication of the *Handbook* and the Memorial Volume :—

HOSPITALITY COMMITTEE

Gordon Douglas, f.f.a.; J. R. Findlay, m.a. (oxon.), j.p., f.r.s.e.; A. Hewat, f.f.a., f.i.a., f.r.s.e.; C. G. Knott, d.sc., *Sec.* r.s.e. (*Convener*).

LOCAL ARRANGEMENTS COMMITTEE

A. G. Burgess, m.a., b.sc., f.r.s.e. (*Convener*); E. M. Horsburgh, m.a., b.sc., assoc. m.inst.c.e., f.r.s.e; C. G. Knott, d.sc., f.r.s.e; D. F. Lowe, m.a., ll.d., f.r.s.e.; D. C. M'Intosh, m.a., d.sc., f.r.s.e.

PROGRAMME COMMITTEE

C. G. Knott, d.sc., f.r.s.e. (*Convener*); Professor R. A. Sampson, m.a., d.sc., f.r.s., f.r.s.e.; Professor E. T. Whittaker, m.a., sc.d., f.r.s., f.r.s.e.

EXHIBITS AND HANDBOOK COMMITTEE

Herbert Bell, m.a., b.sc.; G. A. Carse, m.a., d.sc., f.r.s.e.; D. Gibb, m.a., b.sc., f.r.s.e.; E. M. Horsburgh, m.a., b.sc., assoc. m.inst.c.e., f.r.s.e.; J. R. Milne, d.sc., f.r.s.e.; Professor R. A. Sampson, m.a., d.sc., f.r.s., f.r.s.e.; J. Urquhart, m.a.; Professor E. T. Whittaker, m.a., sc.d., f.r.s., f.r.s.e. (*Convener*).

EDITOR OF HANDBOOK

E. M. HORSBURGH, M.A., B.SC., ASSOC. M.INST.C.E., F.R.S.E.

EDITOR OF MEMORIAL VOLUME

C. G. KNOTT, D.SC., General Secretary Royal Society of Edinburgh.

Shortly before the publication of the *Handbook*, the Committee had to mourn the loss of Mr Urquhart, whose death deprived the Mathematical Department of the University of one of its valued Lecturers.

Strictly speaking, the Congress did not formally open until the afternoon of Friday the 24th of July, when Lord Moulton gave his inaugural address. Visitors and Delegates had, however, begun to appear in Edinburgh some days earlier; and the Executive Committee resolved to open the Exhibition on the morning of the 24th.

This Exhibition was intended in the first place to be an exhibition of (*a*) Napier relics, (*b*) the more outstanding books of logarithms from Napier's day down the centuries, (*c*) the various forms of instrumental aid to calculation, such as slide rules and arithmometers; but it gradually extended its basis, until it included all kinds of graphical methods used in calculation and also models of mathematical surfaces. The nature of the Exhibition will be best understood from the following brief catalogue, taken from the 'Contents' of the *Handbook*.

SECTION A. Napier's Life and Works.

SECTION B. Loan Collection, Antiquarian.

I. Napier Relics. II. Napier's 'Bones' or Numbering Rods. III. Title-pages of Napier's works. IV. Portable Sundials. V. Photographs of early calculating machines. VI. Letters of some early Scottish Mathematicians. VII. Davis Quadrant. VIII. Miscellaneous Exhibits.

SECTION C. Mathematical Tables, chiefly Logarithmic.

I. Historical. II. Sang's Tables. III. Working list of Mathematical Tables. SUBSECTION. IV. Notes on the development of calculating ability.

Section D. Calculating Machines.

I. Calculating Machines described and exhibited: Archimedes; Colt's Calculator; British Calculators; Brunsviga; Burroughs Adding; Comptometer; Layton's Arithmometer; Mercedes-Euklid; Millionaire; Thomas's Arithmometer. II. Automatic Calculating Machines; The Nautical Anti-Differencing Machine. III. Mathematical and Calculating Typewriters: The Hammond and Monarch.

Section E. The Abacus.

Section F. Slide Rules.

Section G. Other Mathematical Laboratory Instruments.

I. Integraphs. II. Integrometers. III. Planimeters. IV. The use of planimeters in Naval Architecture. V. Differentiater. VI. Harmonic Analysis. VII. Tide Predictors. VIII. A mechanical aid in Periodogram work. IX. Conographs. X. Equation Solvers. XI. Instruments for plotting. XII. Precision Pantographs. XIII. Photographic Calculators. XIV. Miscellaneous Instruments.

Section H. Ruled Papers and Nomograms.

I. Ruled Papers: Logarithmic and others. II. Collinear Point Nomograms. III. Computing Forms.

Section I. Mathematical Models:

Models of surfaces, of Regular Solids, of interlacing surfaces; thermodynamic surface of James Thomson and Maxwell's construction of Willard Gibbs's Surface; linkages; semi-regular polyhedra; projection model of the 600-cell in space of four dimensions; parallel motion models; pantographs; plastographs and anaglyphs.

Section K. Portraits and Medals.

Section L. Miscellaneous and Late Exhibits.

Section M. List of Contributors and Exhibitors, which is here reproduced with a few corrections.

Adams, A. C., A.M.I.M.E., 1 Old Smithhills, Paisley, F (4).

Anderson, Brigadier-General F. J., 4 Treboir Road, Kensington, London, S.W., F (3).

Andoyer, H., Professeur à la Faculté des Sciences de l'Université de Paris, Membre du Bureau des Longitudes, Paris, C, I.

Archimedes and Colt Calculating Machines, 4 Albert Square, Manchester (Agent, Alex. Angus, 61 Frederick Street, Edinburgh), D, I. (1).

Ball, W. W. Rouse, M.A., Fellow of Trinity College, Cambridge, K, I. and II.

Barkla, Charles G., D.Sc., F.R.S., F.R.S.E., Professor of Natural Philosophy, University of Edinburgh, B, I. (5).

Baxandall, D., South Kensington Museum, London, W., F (1).

Beghin, Auguste, L, XI.

Bell, Herbert, M.A., B.Sc., Assistant in Natural Philosophy, University of Edinburgh, C, III. (a).

Bennett, G. T., M.A., F.R.S., Fellow of Emmanuel College, Cambridge, I, IX. and X.

Blackmore, Mrs., Forden House, Moretonhampstead, C, IV. (1).

British Calculators 'Brical,' Invicta Works, Belfast Road, Stoke Newington, London, N., D, I. (3).

Brown, Professor A. Crum, M.D., D.Sc., LL.D., F.R.S., F.R.S.E., 8 Belgrave Crescent, Edinburgh, **I**, (*a*) and I.-VII.
Brunsviga Calculating Machines, Wellington Chambers, 46 Cannon Street, London, E.C. (G. M. Müller, Sales Manager), **D**, I. (4).
Burgess, A. G., M.A., F.R.S.E., 64 Strathearn Road, Edinburgh, **L**, X.
Burroughs Adding and Listing Machine (E. Hawkins, Sales Manager, Cannon Street, London, E.C.), **D**, I. (5).

Carse, George A., M.A., D.Sc., F.R.S.E., Lecturer in Natural Philosophy, Univ. of Edinburgh, **G**, II. (*a*), III. (*a*), VI. (*a*).
Colt's Calculators, 4 Albert Square, Manchester, **D**, I. (2).
Comptometer (see Felt and Tarrant), **D**, I. (6)
Coradi, G., Zürich, **G**, II. (1), III. (1), XI. (1) and (2).

Dantzig, Town Library of, **C**, I.
Davis, John, & Sons, All Saints Works, Derby, **F**, (16).
Dickstein, Professor S., Warsaw, **L**, III.
Dunlop, Mrs. Mercer, 23 Campbell Avenue, Murrayfield, Edinburgh, **B**, II. (3).

Education, Board of, London, **B**, V.
Erskine-Murray, J. R. See Murray.
Esnouf, Auguste; A.C.G.I., Port Louis, Mauritius, **F** (9).
Euklid Calculating Machine (see Mercedes-Euklid).
Evans, Lewis; Russells, near Watford, **B**, II. (1); **F** (2).

Faber, A. W., **F** (17).
Felt & Tarrant Manufacturing Co., Imperial House, Kingsway, London, W.C., **D**, I. (6).
Fergusson, John C., M.Inst.C.E., **L**, XIII.
Findlay John R., M.A., D.L., J.P., F.R.S.E., 27 Drumsheugh Gardens, Edinburgh, **B**, IV.; **C**, I.; **L**, I.
Forrester, Miss Catherine, 30 Snowdon Place, Stirling, **B**, I. (8).

Gibb, David, M.A., B.Sc., F.R.S.E., Lecturer in Mathematics, University of Edinburgh, **G**, VIII., IX., X. (*a*).
Gibson, G. A., M.A., LL.D., F.R.S.E., Professor of Mathematics, University of Glasgow, **A** (*a*).
Gifford, Mrs. E., Oaklands, Chard, **L**, VI.
Gregorson, A. M., W.S., Ardtornish, Colinton, **B**, II. (2).

Hammond Typewriter Co., 50 Queen Victoria Street, London, E.C., **D**, III. (1).
Harvey, Major W. F., I.M.S., Director Pasteur Institute of India, Rasauli, **E**, (2).
Henderson, Adam, F.S.A. (Scot). The Library, University of Glasgow, **B**, VIII.
Hewat, Archibald, F.F.A., F.I.A., F.R.S.E., 13 Eton Terrace, Edinburgh, **K**, III. (1).
Hilger, Adam (see Millionaire Calculating Machine).
Hippisley, Colonel R. L., C.B., R.E., 106 Queen's Gate, South Kensington, London, S.W., **K**, VIII.
Horsburgh, Ellice M., M.A., B.Sc., Assoc. M.Inst.C.E., F.R.S.E., Lecturer in Technical Mathematics, University of Edinburgh, **D**, I. (10); **F** (10); **G**, II. (2); **H**, I.; **I**, XXIV.-XXVI.; **K**, III. (2).
Horsburgh, the Rev. Andrew, M.A., Lynton, St. Mary Church, Torquay, **B**, VII.
Hudson, T. C., B.A., H.M. Nautical Almanac Office, 3 Verulam Buildings, Gray's Inn, W.C. See Burroughs Adding Machine, **D**, II. (1).

Jardine, W., M.A., B.Sc., 40 Albion Road, Edinburgh, **D**, I. (8); **H**, I. (1).

Kelvin, Bottomley & Baird, 16-20 Cambridge Street, Glasgow, **G**, VII. (2).
Knott, Cargill G., D.Sc., General Secretary R.S.E., Lecturer on Applied Mathematics in the University of Edinburgh, and formerly Professor of Physics in the Imperial University of Tokyo, Japan, **E** (*a*) and (1); **C**, II.; **I**, VII.

Langley, Edward M., M.A., Bedford Modern School, **I**, Subsections I. and II.
Layton, C. & E., 56 Farringdon Street, London, E.C., **D**, I. (7).
Lilly, W. E., M.A., M.A.I., D.Sc., M.I.C.E., Ireland, Engineering School, Trinity College, Dublin, **F** (5).

Macdonald, W. R., F.F.A., 4 Wester Coates Avenue, Edinburgh, **B**, III.; **K**, III. (6).
Maclean, J. M., B.Sc., 8 Forth Street, Edinburgh, **F** (13).
Mathematical Laboratory, University of Edinburgh, Groups of Exhibits in Sections **C**, **D**, **G**, **H**, **I**, **K**.
Mercedes-Euklid Calculating Machine (F. E. Guy, Agent, Cornwall Buildings, 35 Queen Victoria Street, London, E.C.), **D**, I. (8).
Meteorological Office, London, S.W., **H**, III (2).
Miller, Professor John, M.A., D.Sc., F.R.S.E., Royal Technical College, Glasgow, **G**, I. (1).
Millionaire Calculating Machine (Adam Hilger), 75A Camden Road, London, N.W., **D**, I. (9).
ilne, J. R., D.Sc., F.R.S.E., Lecturer in Natural Philosophy, University of Edinburgh, **C**, III.; **L**, V.
Monarch Typewriter Company, 165 Queen Victoria Street, London, E.C., **D**, III. (2).
Muirhead, R. F., B.A., D.Sc., 64 Great George Street, Hillhead, Glasgow, **F** (7), **G**, X. (1).
Murray, J. R. Erskine-, D.Sc., F.R.S.E., M.I.E.E., 4 Great Winchester Street, London, E.C., **G**, V.
Murray, T. Blackwood, Esq., Heavyside, Biggar, **B**, I. (6).

Napier and Ettrick, the Right Hon. Lord, Thirlestane, Selkirk, **B**, I. (7).
Napier, Archibald Scott, Esq., Annelshope, Ettrick, Selkirk, **B**, I. (1) and (2).
Napier, Sir Alexander L.. 56 Eaton Place, London, S.W., **L**, XIV.
Napier, Miss, 74 Oakley Street, Chelsea, London, S.W., **B**, I. (3).

Observatory, Royal, Blackford Hill, Edinburgh, **C**, I., various.
d'Ocagne, M., Professeur à l'École Polytechnique, Paris, **H**, II.
Ott, A., Kempten, Bavaria, **G**, III. (2); XI. (3); XII.

Pascal, Professor Ernesto, University of Naples, **G**, I.
Peddie, Professor W., D.Sc., F.R.S.E., University College, Dundee, **L**, IX.
Presto Calculating Machine (see Taussig).

Robb, A.M., Department of Naval Architecture, University of Glasgow, **G**, IV.
Robb, John, St. Cyrus, 4 King's Park Road, Mount Florida, Glasgow, **B**, II. (4).
Roberts, Edward, I.S.O., F.R.A.S., Park Lodge, Eltham, **G**, VII.
Robertson Rapid Calculating Co., 38 Bath Street, Glasgow, **G**, XIV. (1).
Royal Society of Edinburgh, **C**, II.

Sampson, Professor Ralph A., M.A., D.Sc., F.R.S., F.R.S.E., Royal Observatory, Blackford Hill, Edinburgh, **C**, I.
Schleicher & Schüll, Düren, Rheinland, Germany, **H**, I. (1) and (2).
Science Museum, South Kensington, London, **B**, V.
Smith, Professor D. Eugene, Teachers' College, Columbia University, New York, **C**, IV. (2) and (3).
Smith, George, M.A., Headmaster of Merchiston Castle, Edinburgh (Illustrations, 3),
Smith, Professor R. H., **L**, VII.
Smith, W. G., M.A., Ph.D., Lecturer in Psychology, University of Edinburgh, **C**, IV.
Sommerville, D. M. Y., D.Sc., F.R.S.E., Lecturer in Mathematics, University of St. Andrews, **I**, XV. and XVI.
Spencer, John, F.I.A., 33 St. James's Square, London, S.W., **C**, I.
Spicer, George (see T.I.M. and Unitas Calculating Machine Co.).
Stainsby, Henry, British and Foreign Blind Association, 206 Great Portland Street, London, W., **C**, IV. (4).
Stanley, W. F. & Co., Ltd., Scientific Instrument Makers, Glasgow, **F** (15).
Steggall, J. E. A., M.A., F.R.S.E., Professor of Mathematics, University College, Dundee, **I**, XI.-XIV.
Stokes, G. D. C., M.A., D.Sc., Department of Mathematics, University of Glasgow, **F** (*a*) and Section.
Stuckey, J. J., M.A., A.I.A., **L**, XII.

Tate Arithmometer, **D**, I. (10).
Taussig, Dr. Rudolf, 8 Wuerthgasse, Wien **F** (6).
Thomas de Colmar Arithmometer, **D**, I. (10).
Thornton, A. G., Paragon Works, King Street W., Manchester, **F** (18).

T.I.M. and Unitas Calculating Machine Co., 10 Norfolk Street, Strand, London, W.C., **L**, IV.

Tweedie, Charles, M.A., B.Sc., F.R.S.E., Lecturer in Mathematics, University of Edinburgh, **G**, I.

University of Edinburgh, Departments of (1) Engineering, **F** (12); (2) Natural Philosophy, **B**, I. (5); (3) Library, **C**, I., various; (4) Mathematical Laboratory; Mathematical Tables; Calculating Machines; Calculating and Curve-Tracing Iustruments; Computing Forms; Models; and Portraits.

University of Glasgow, (1) Department of Electrical Engineering, **F** (11); (2) Library, **C**, I.

University College, London, **C**, I.

Urquhart, John, M.A., B.A. (Cantab.), late Lecturer in Mathematics, University of Edinburgh, **G**, II. (*a*), III. (*a*), VI. (*a*).

Warden, J. M., F.F.A., Scottish Equitable Life Assurance Co., 28 St. Andrew Square, Edinburgh, **F** (8).

Watkins, Alfred, F.R.P.S., Imperial Mills, Hereford, **G**, XIII.

Wedderburn, E. M., M.A., D.Sc., F.R.S.E., 7 Dean Park Crescent, **L**, VIII.

Whipple, F. J. W., M.A., Superintendent, Instrument Division, Meteorological Office, South Kensington, London, S.W., **D** (*a*).

Whittaker, Edmund Taylor, M.A., Sc.D., F.R.S., F.R.S.E., Professor of Mathematics, University of Edinburgh (see University of Edinburgh Mathematical Laboratory).

Woodward, C. J., The Lindens, 25 St. Mary's Road, Harborne, Birmingham, **L**, I. and II.

Young, A. W., M.A., B.Sc., 14 Dudley Avenue, Leith, **H**, II.

The *Handbook* contains three portraits of John Napier, one of Charles Babbage, and one of Edward Sang, also a view of Merchiston Castle. Many of the articles are profusely illustrated.

Members when they enrolled were presented with the *Handbook* to the Exhibition, and a *Guide-Book* to Edinburgh, with an introduction referring to the arrangements for the Tercentenary. They also received from Mr George Smith, M.A., Head Master of Merchiston Castle School, a pamphlet written by him, containing an account of the Castle in which Napier lived and calculated his logarithms. This sketch forms the third article in the Memorial Volume.

By courtesy of the President and Committee of the Students' University Union, all Members of the Congress enjoyed full privileges of membership of the Union during the time of the Congress.

The numerous exhibits were laid out in Examination Hall A of the University of Edinburgh. The Office, Reception Room and Tea Room were in the neighbouring Examination Hall B.

The Opening Ceremony of the Tercentenary Celebration took place on Friday the 24th of July, at 4.30 P.M., in the Debating Hall of the Students' Union of the University of Edinburgh.

The Inaugural Address was delivered by Lord Moulton, who was accompanied to the platform by the Right Honourable the Lord Provost of Edinburgh and a selection of foreign and other Delegates.

Lord Provost Inches presided.

The Lord Provost in introducing Lord Moulton said:—'This is not the occasion to refer to all the brilliant episodes in Lord Moulton's distinguished career. The ordinary man thinks of him as a learned Judge, dispensing justice and equity in the House of Lords; but we who are met to do honour to the inventor of logarithms look deeper. It is not every Lord of Appeal who can do justice to the memory of John Napier of Merchiston. But in Lord Moulton we have one whose first intellectual love was mathematics, and no lover of mathematics can fail to admire the logarithm. In his undergraduate days, Lord Moulton gained every mathematical prize within his reach, and he finally completed his student's career as Senior Wrangler and First Smith's Prizeman. He remained a few years at Cambridge as fellow and lecturer, and then, attracted by the wider human activities of law and politics, he forsook the quiet cloisters by the Cam, and pitched his tent by the swirling Thames. One permanent monument marks his stay at Cambridge. As editor of the second edition of Boole's *Calculus of Finite Differences,* he practically re-wrote the larger part of the book. This subject, it should be noted, has close affinity with the construction of logarithmic as well as other kinds of mathematical tables. Thus historically, as authors, John Napier and Lord Moulton join hands. It is worthy of mention that even in the engrossing duties of a steadily growing legal practice Lord Moulton found time for research work in science. In conjunction with the late Sir William Spottiswoode, he made investigations in the electric discharge through gases. Their results were published by the Royal Society of London, and formed an important link in the development of modern views on the nature of electricity. These facts show how singularly well fitted Lord Moulton is to open a congress of mathematicians, who have been brought together to do homage to the memory of our great Scotsman and citizen, the sage of Merchiston.'

Lord Moulton then delivered his Address on 'The Invention of Logarithms, its Genesis and Growth.' This Address forms the first article in the present volume.

While Lord Moulton was delivering his address the following telegrams of congratulations were received:

1. From Professor VOLTERRA, Ariccia, near Rome: Unite celebration tercentenary immortal Napier regret that unforseen circumstances do not allow me to be present.

2. From Comandante VERDE, Spezia: Riverente augurio osservatorio meteorologico Spezia.

3. From Rector KOSTANEKI, Krakow: From afar the Senatus Academicus of the Jagellonian University of Cracow send a message of warmest congratulations to the President and Fellows of the Royal Society of Edinburgh on the occasion of the Tercentenary Celebration of the publication of John Napier's splendid discovery.

4. From President SINTSOF, Mathematical Society, Kharkoff: Société Mathématique de Kharkoff s'associe à la célébration du tricentenaire du fameux inventeur des logarithmes. Elle pria Stekloff la représenter.

5. From President TARNOWSKI and Secretary ULANOWSKI, Krakow: The President and Council of the Imperial

Academy of Sciences, Cracow, desire to present heartfelt greetings to the President and Fellows of the Royal Society of Edinburgh on the occasion of the Napier Tercentenary Celebration.

6. From PRINCE GALITZIN, Petersburg: The Director of the Central Physical Observatory of St. Petersburg regrets exceedingly not to have been able to be present at the celebration of Napier's Tercentenary, and desires to offer his sincere congratulations and hopes that the celebration will meet with a success worthy of so distinguished association.

7. From Rektor ZINDLER, Innsbruck: Royal Society, George Street, Edinburgh, In treuem Gedenken an den beruehmten Verfasser Mirifici Canonis Logarithmorum beglueckwuenscht die Universitaet Innsbruck die Royal Society seltene feier Erben Rektor ZINDLER dekan.

8. From Rector ELATARSKI, Sofia: Edinburgh Royal Society, Senate of Sofia University sends hearty greetings to members of Congress celebrating Tercentenary John Napier's Scientific Work.

9. From Rektor PLANCK, Berlin: Bei der Dreihundersten Wiederkehr des Tages an welchen John Napier der Welt die Logarithmen schenkte nimmt die Universitaet Berlin mit herzlichen Glueckwuenschen teil an der heutigen Erinnerungsfeier.

Dr Knott then announced the names of the Delegates representing various Universities, Academies, Societies, and Corporations.

Four of the Delegates briefly addressed the meeting, two in French, one in German, and one in English, namely: Professor Andoyer, Paris; Professor d'Ocagne, Paris; Professor Bauschinger, Strassburg; and Professor D. Eugene Smith, New York.

Emeritus-Professor Geikie, President of the Royal Society of Edinburgh, proposed a vote of thanks to Lord Moulton for his admirable address.

In the evening the Members of the Tercentenary and their Hosts and Friends attended a brilliant Reception in the Usher Hall, given by the Lord Provost, the Magistrates and the Town Council of the City of Edinburgh. A special interest was attached to this Reception as being the first to be held in the new Hall. Attended by the City Halberdiers, the Lord Provost and his Colleagues, wearing their robes of office, took up a position on the platform in front of the organ, and received the guests as they passed across the platform from the Lord Provost's left.

The guests then took seats in the various parts of the Hall to listen to the programme of music which had been provided. Refreshments were served in the ample corridors.

Saturday the 25th of July 1914

The Napier Tercentenary Congress met in the University on Saturday, July 25, at 9.30 A.M., for the reading and discussion of papers bearing directly on Napier's work. Professor Hobson of Cambridge was voted to the Chair.

About 11.30 A.M. the Members adjourned for half an hour to the Reception Room for tea and coffee refreshments and an informal Conversazione; and the sederunt came to an end about half-past one.

The following papers were communicated:

1. Logarithms and Computation. By J. W. L. GLAISHER, M.A., Sc.D., F.R.S., Trinity College, Cambridge.
2. The Law of Exponents in the Works of the Sixteenth Century. By DAVID EUGENE SMITH, Ph.D., LL.D., Columbia University, New York.
3. Algebra in Napier's Day and alleged Prior Inventions of Logarithms. By FLORIAN CAJORI, Ph.D., Sc.D., Colorado Springs University, U.S.A.

4. Napier's Logarithms and the Change to Briggs's Logarithms. By George A. Gibson, M.A., LL.D., F.R.S.E., University of Glasgow.
5. Introduction of Logarithms into Turkey. By Salih Mourad, Lieutenant in the Turkish Navy.
6. The First Napierian Logarithm calculated before Napier. By Giovanni Vacca, Royal University of Rome.
7. The Theory of Napierian Logarithms explained by Pietro Mengoli (1659). By Giovanni Vacca, University of Rome.
8. Napier's Rules and Trigonometrically Equivalent Polygons. By D. M. Y. Sommerville, M.A., D.Sc., F.R.S.E., University of St. Andrews (now of Wellington University, New Zealand).

To these should be added the two following communications which were, in the absence of their Authors, taken as read.

John Napier of Merchiston. By P. Hume Brown, M.A., LL.D., University of Edinburgh.

A Short Account of the Treatise *De Arte Logistica.* By J. E. A. Steggall, M.A., F.R.S.E., University of St. Andrews at Dundee.

These communications, along with an architectural and historical sketch of Merchiston Castle, by George Smith, M.A., Head-master of Merchiston School, form the first eleven articles in the present volume.

During the afternoon of Saturday the Governors of Merchiston Castle School invited the members of the Tercentenary Congress to a Garden Fête in the grounds of the old castle of Merchiston. Although outside the city walls in John Napier's day, Merchiston Castle is now well within the municipal boundaries of Edinburgh.

The Members of the Congress were conducted in small parties over the castle, and shown the room where it is believed Napier carried out his laborious calculations.

A large group of the Delegates and their friends was photographed with the castle in the background. The Secretary, in proposing a vote of thanks to the Governors for their hospitality, congratulated the Head Master on his recent appointment as Master of Dulwich College.

On the evening of Saturday, the General Committee entertained the Members at an informal Reception and Conversazione in the University Union. The guests were received by Dr and Mrs Knott and Mr and Mrs Burgess. Light refreshments were served in the Dining Hall, and a programme of music, both instrumental and vocal, was provided in the Debating Hall, mainly by students of the University.

Sunday the 26*th of July* 1914

On the afternoon of Sunday a Memorial Service was held in St. Giles Cathedral. A very large representation of the Delegates and other Members of the Congress gathered under the Gothic arches of this historic Church, above whose massive pillars wave the old and tattered flags of Scotland's heroic regiments.

On the south demi-pillar at the east end of the choir there is scupltured the Napier Shield, pointing to a close association of the family of Napier with the Cathedral. The following quotation from the Rev. Dr J. Cameron Lees' history *St. Giles, Edinburgh, Church, College, and Cathedral, from the earliest Times to the Present Day,* brings out this association very clearly :—

'The arms on this shield [on the south demi-pillar at the east end of the choir], a saltier engrailed cantoned between four roses, have been ascribed to Napier of Merchiston, who was Provost of Edinburgh in 1457, and to Isabella, Duchess of Albany and Countess of Lennox, a daughter of Duncan, Earl of Lennox, and the wife of Murdoch, Duke of Albany. . . .

'The Napiers had even a closer connection with St. Giles than what might seem to be indicated by this shield. They appear to have had a burial vault under the arched recess on the north wall of the choir, below the second window from the east, and it is said that Sir Alexander Napier was buried there. The recess has been in the wall since the middle of the fifteenth century, and was at one time fringed with finely carved crockets representing oak leaves. Following the example of the Romans, who rewarded the saving of life by the presentation of a crown of oak leaves, it is permissible to believe that the tomb of the gallant defender of Queen Joanna was adorned with a similar chaplet carved in stone. On the outside of the wall there is a tablet with an inscription setting forth that it marks the family sepulchre of the Napiers, and the tablet was there in 1753, when Maitland published his *History of Edinburgh*, and probably long before that time. The coat of arms on the shield above the tablet is composed of the arms of the Napiers of Merchiston and the Napiers of "Wrychtishousis," two separate and distinct families, and probably records in this heraldic form the marriage which took place, about 1513, between a daughter of the latter house and a son of the former. . . .

'On the outside wall of the choir, on the north side, there is a stone tablet with the following inscription :

SEP

FAMILIAE NAPERORV INTERIVS

HIC SITVM

EST.

This inscription is surmounted by the arms and crest of Napier, with the Wrychthousis shield. The tablet is evidently connected with the burying-place of the ancient family of Napier of Merchiston, who, in old times, were closely associated with the church. Whether this tablet occupies the original site where it was first placed is open to question. At one time it is mentioned as having been "in front of the church," at another as occupying a position inside ; and Arnot, in his *History of Edinburgh*, says that " in different quarters of the church there are monuments of the celebrated Lord Napier of Merchiston." These monuments have disappeared, and the slab on the outside of the choir was inserted where it is by Mr Burn. There is evidence to show that the Napiers of Merchiston long buried in St. Giles, but it is not certain whether the celebrated member of the family, Baron Napier, the inventor of logarithms, who died in 1617, was laid there or not. . . .'

It does not appear, however, that John Napier of logarithmic fame had any connection with St. Giles. He was an Elder in the Parish Church of St. Cuthbert's, and was buried in St. Cuthbert's Churchyard.

It was appropriate, therefore, that the Memorial Service in the National Cathedral should be conducted by the Minister of St. Cuthbert's, the Rev. R. H. Fisher, D.D. The Scripture Lessons were taken from the Book of Job, chap. xxxviii, verses 25-41, and from the Gospel according to St. Matthew, chap. xxv, verses 14-30, and were read by Dr C. G. Knott, the Secretary of the Tercentenary Congress.

Dr Fisher preached from the Text 'So teach us to number our days that we may apply our hearts unto wisdom.' —Ps. xc, 12. The sermon was as follows:—

In a memorable speech in 1825, that great Scotsman, Thomas Chalmers, reproached himself over the period of his life when he was so much occupied with merely mathematical interests that he neglected the realities of time and eternity. 'What, Sir,' he asked, 'is the object of mathematical science? Magnitude and the proportions of magnitude. But then, Sir, I had forgotten two magnitudes. I thought not of the littleness of time. I recklessly thought not of the greatness of eternity.'

Because the illustrious John Napier, whom the whole scientific world is honouring in these days, did not forget those two magnitudes and was a reverent and pious man, it is fitting that remembrance should be made of him, not only in the Schools where his supreme and commanding genius for mathematics is recognised, but also in the House of God.

In the Church we reverence science and give thanks for all its victories. The followers of Him who said, 'I am the Truth,' can never believe that it is in the interests of truth that any one truth should be refused a welcome or should be neglected or denied. But here also we are brought into contact with things intangible and immeasurable—with ideals of duty, with heroism and saintliness, with the brooding presence of the Infinite, with the hopes and fears which the unknown future inspires.

Beside such sublime realities we adjust our judgments of greatness. We feel that there is something more satisfactory in the account which was given of Napier by his friend and minister, my celebrated predecessor in the charge of St. Cuthbert's, Robert Pont, than even in the tremendous testimony of David Hume. In the seventh chapter of his *History*, Hume refers to Napier as 'the person to whom the title of A Great Man is more justly due than to any other whom his country ever produced.' That is a wonderful tribute, and true at a time when Burns and Scott had not been born. Yet within this place of prayer to-day we welcome, with even deeper appreciation of its value, Robert Pont's description of Napier. Pont wrote of him as 'that faithful servant of Christ, my honoured and surpassingly learned friend, John Napier.' Napier would have aspired to the title 'Servant of Christ,' rather than to any other earthly fame.

It is not necessary, in order to think of Napier as an eminent Christian, to extol him in the exaggerated language of his biographer as 'the first and greatest theologian' of the Church of Scotland. Yet Napier brought to his excursion into theology rare qualities of mind and character. It is not permissible to condemn it with the term 'puerile' which one critic used concerning it, or to describe it with Sir Walter Scott as 'a waste of time.' The truth is that an examination of Napier's theological book, which he called

A Plain Discovery of the whole Revelation of St. John, discovers gifts as uncommon and as suggestive of mastery as even his mathematical inventions.

Besides that delight in tracing the hand of Providence in history which has ever been the mark of Scottish piety, the book shows a spirit of fairness in controversy very unusual at Napier's time: there is throughout it an evident anxiety over scholarly accuracy; there is, along with ingenious speculation, the modesty which accompanies true learning and capacity. Of course, the book is only of historical interest for theology now, when the whole subject of Apocalyptic is regarded from an altered point of view. But it can never be less than useful to witness the effort of an adventurous and fertile mind to reach true conclusions along lines manifestly honest and laborious. Napier's mistakes in the exposition of Scripture were no bigger than Newton's, and his book exhibits a spirit equally sincere and Christian. A man so virile and independent may fall into error. But the impression left by his candour and courage and love of truth is not the less deep and lasting.

Napier's whole life was profoundly modified by influences which, if they cannot always be described as religious, were at least ecclesiastical. The year 1550, in which he was born—the year after John Knox escaped from the French galleys—is sometimes spoken of as the date of the Scottish Reformation. It was amid the disturbing environment of that great cataclysm of the religious life of the country that the philosopher's powers came to maturity.

At an extraordinarily early age he betrayed an enthusiasm for the Protestant cause. The only glimpse we get of his boyhood discloses him as already a student of Scripture and a controversialist against the Papacy. Later he was to become an office-bearer in St. Cuthbert's parish church, where his family were among the chief heritors. It was as an elder in that church that he was sent by the Presbytery of Edinburgh, in the year 1588—the year of the Spanish Armada—to represent them in the General Assembly. The prominence which he subsequently reached in the chief court of the Scottish Church is illustrated by the fact that in 1593 he was chosen by it to be one of four Commissioners to lay before King James VI. certain grievances and pleas—a task requiring no little courage and tact. It is all to the credit of the philosopher that he was willing in such ways to serve his Church.

Public spirit in the largest sense should be a mark of the Christian character. This also Napier showed. He came of an ancient Scottish stock with whom public service had been an honourable tradition. Three of his ancestors had been Provosts of this city. His father was one of the four Justice-deputes of Scotland. He himself was not so absorbed in mathematical inquiries as to shrink from his share of concern for the common weal. The national danger from Spain led his restless brain to the invention of warlike devices for defence. The barrenness of his native country and the backwardness of its agriculture turned his mind to plans for the fertilising of the soil. Thus, out of his meagre biography—as we get it in the discursive and provoking yet delightful pages of Mark Napier—there arises a personality not less commendable for unselfish service to Church and country, and for the more intimate graces of a Christian character, than for consummate intellectual power.

The merely negative fact that Napier took no part in the coarse intrigues and struggles of his time is itself a testimony to his essential goodness. Not likely to have thought favourably of either the religion or the policy of Mary, Queen of Scots, he yet dissociated himself from the persecution of that unhappy but fascinating person. Between him and the Scottish barons there must have been as little sympathy as between Daniel and the lions in his den, and we learn of no co-partnery between him and them in any scheme of greed or ambition. Crime and cruelty and judicial torture were the outward accompaniments of his life: yet through such grim surroundings Napier passed—a kindly, sensitive, domestic, studious, godly man.

It is no reproach against Napier's character that he dabbled in the occult, and seems even to have believed that he himself possessed mysterious powers. Men whose preoccupation is with the exact sciences have often, before and since his day, been fascinated by the Borderland. It must be remembered that Napier's speculations on such subjects were restrained in comparison with the extravagances of popular belief. Such as they are, they bring him as little discredit as comes to a modern man of science from interest in the methods and results of the Society for Psychical Research. There are stories extant which seem to show that Napier's sense of humour was not unaffected by the popular estimate of his powers as a warlock. Where humour is, there can hardly be any gross superstition of which a religious spirit should be ashamed.

Such, then, was the man whose invention of logarithms is declared by competent judges to place him in the foremost rank of those who have contributed to the sum of human knowledge—beside Copernicus, Kepler, Newton. Those whose concern is with astronomy or geodesy or navigation, or tables of insurance, testify to the relief which his invention has brought to their sciences. It is well that he should be known also as the author of a study of Scripture as original as his mathematical achievement, and as an upright and godly man.

Scotland ought to be proud of Napier. Yet it is a melancholy fact that Scottish history is almost silent regarding him. Hill Burton gives a page and a half of his eight volumes to the record of his mathematical work, and Hume Brown gives four lines of his *History of Scotland* in three volumes. But in Tytler's nine volumes, and Andrew Lang's four, his name is not even mentioned. Yet Napier lightened a dull page in our nation's intellectual life. George Buchanan's reputation, indeed, was high at the time among the educated men of Europe—higher than Shakespeare's. But in comparison with the galaxy of genius that shone in England in those days of Elizabeth and James, there are only two Scottish names that have come to great renown —Buchanan and Napier. We do well to honour the greater and better of the two.

Edinburgh has the fullest reason for associating herself with the celebration of this tercentenary. It is true that Napier's ancestral home of Merchiston was not then as now embedded among her streets; the gaunt fortalice stood, forbidding and solitary, among neighbouring fields, parted by a moor from the little town on the Castle ridge. But Napier's feet must often have trodden the streets of Edinburgh. And Sunday after Sunday he worshipped in St. Cuthbert's parish church, which stood then, as for eight hundred years before, under the shadow of the Castle rock. Beside St. Cuthbert's church they buried him in 1617—the year after Shakespeare died. 'Near this spot,' so runs the inscription upon a monument in the church tower, 'was laid the body of John Napier of Merchiston, who gained for himself the imperishable memory of future ages by his wonderful discovery of logarithms.' Edinburgh has had only one greater citizen—and he great in walks so different that it is almost ludicrous to compare Napier and Scott.

The frontispiece of Napier's book *A Plain Discovery*, bears a heraldic device representing the arms matrimonial of Scotland and Denmark in conjunction, a compliment to King James and his queen. With some reference to that alliance and in deepest reverence of spirit, Napier wrote below the picture these affecting words, characteristic alike of his courage and of his profoundly pious mind : 'In vain are all earthly conjunctions unless we be heirs together, and of one body, and fellow-partakers of the promises of God in Christ by the evangel.' Could there be a more moving suggestion of the solemn thought of Chalmers which I quoted when I began? In presence of all the knowledge which the human mind can gain and master there is one truth which stands supreme above all in solitary significance —the dependence of man's feeble spirit upon the mercy of Almighty God. The final triumph of mathematics has been reached when we know realisingly the littleness of time and the greatness of eternity. Lord, 'so teach us to number our days that we may apply our hearts unto wisdom.'

Monday the 27*th of July* 1914

The Tercentenary Congress met on Monday morning at 9.30 A.M. in the mathematical class-rooms of the University. As on the Saturday, there was a short adjournment for conversation and light refreshments about eleven o'clock. Professor David Eugene Smith of New York was voted to the Chair in the earlier part of the sederunt; and Dr Glaisher and Major MacMahon acted as Chairmen during the later hours.

The following papers were communicated:

1. Nouvelles Tables trigonométriques et logarithmiques fondamentales. By Professor H. ANDOYER, Paris Observatory.
2. Formeln und Rechenschema für die Entwickelung einer Funktion zweier Variabeln nach Kugelfunktionen. By Professor BAUSCHINGER, Observatory of Strasburg.
3. Tables numériques et Nomogrammes. By Professor M. D'OCAGNE, l'École Polytechnique, Paris.
4. Sur l'Origine des Machines à Multiplication directe. By Professor D'OCAGNE, l'École Polytechnique, Paris.
5. New Tables of Natural Sines to Seconds of Arc. By Mrs E. GIFFORD, Chard, Somerset.

6. The Arrangement of Mathematical Tables. By J. R. Milne, D.Sc., F.R.S.E., University of Edinburgh.
7. Note on Critical Tables. By T. C. Hudson, B.A., H.M. *Nautical Almanac* Office, London.
8. Graphical Methods in Crystallography. By A. Hutchinson, M.A., Ph.D., Pembroke College, Cambridge.
9. On Computing Logarithms, Reciprocals, and Gaussian Logarithms by simple Addition. By William Schooling, F.R.A.S., London.
10. Extension of Accuracy of Mathematical Tables by Improvement of Differences. By W. F. Sheppard, Sc.D., LL.M.
11. Approximate Determination of the Functions of an Angle and the Converse. By H. S. Gay, United States of America.
12. Percentage Unit Tables and Theodolite. By J. C. Fergusson.

Professor Nielsen of Copenhagen gave an account of the Danish Translation of Napier's *Rabdologia.*

The following papers were taken as read:

13. Dr Edward Sang and his Logarithmic Calculations. By the Secretary, C. G. Knott, D.Sc., General Secretary R.S.E.
14. On a possible Economy of Entries in Tables of Logarithmic and other Functions. By Professor J. A. E. Steggall, M.A., F.R.S.E., University of St. Andrews at Dundee.
15. How to Reduce to a Minimum the Mean Error of Tables. By A. K. Erlang, M.A., Copenhagen, M.Inst.E.E.
16. Notes on Logarithms. By Artemas Martin, LL.D., Washington, United States of America.
17. A Method of Finding without the use of Tables the Number corresponding to a given Natural Logarithm. By Artemas Martin, LL.D., Washington, United States of America.

M. Albert Quiquet, Secrétaire générale de l'Institut des Actuaires français, had intended to be present at the Tercentenary Congress and to read a paper 'Probabilités viagères

—sur un Critérion logarithmique de M. Goldziher et sur son Extension.' He was, however, prevented from coming at the last moment.

During the afternoon the President and Council of the Royal Society gave the farewell Reception to the Delegates and Members of the Tercentenary Congress. The Reception was held in the Royal Society House, George Street, Edinburgh; and Professor James Geikie and Mrs Geikie received the guests in the Reception Room.

Dr Sang's thirty-seven manuscript volumes of logarithms and trigonometrical functions were laid out in the Library and attracted much attention from the visitors.

Many of the Members of the Tercentenary Celebration continued for a few days to take part in the Mathematical Colloquium organised by the Edinburgh Mathematical Society.

Most of the visitors from Continental countries had purposed touring through the Highlands of Scotland; but the ominous conditions preceding the outbreak of the Great War compelled them to return as quickly as possible to their native lands.

The Napier Tercentenary Celebration will go down to history as the last International Congress before the momentous European international struggle which began in August of 1914.

A number of Addresses congratulating the Royal Society of Edinburgh on the occasion of the Napier Celebration were received from Academies and Universities. These are given on the following pages.

CONGRATULATORY ADDRESSES

FROM THE ROYAL DANISH ACADEMY OF SCIENCES

TIL THE ROYAL SOCIETY OF EDINBURGH

HÆDRENDE Mindet om SKOTLANDS berömte Sön sender DET KONGELIGE DANSKE VIDENSKABERNES SELSKAB THE ROYAL SOCIETY OF EDINBURGH sin Lykönskning i Anledning af Tre Hundrede Aars Festen for Offentliggörelsen af JOHN NAPIER'S *Logarithmorvm Canonis Mirifici Descriptio.*

Ved sit Livs Storværk, Logarithmernes Opfindelse, har JOHN NAPIER ikke blot fört tidligere Tiders famlende Forsög paa at lette Multiplikationsarbejdet til en endelig og fuldstændig Afslutning, men han blev tillige ved for förste Gang at bestemme en ny, transcendent Funktion ad infinitesimal Vej en Forgænger for Newton, til hvis Fluxionstheori de förste Spirer kunne findes i NAPIER'S *Constrvctio.* Med fuld Föje hyldes da NAPIER af Efterverdenen som en af Videnskabens Banebrydere.

Ikke mindre har han ved det inderlige Baand, hvormed han sammenknytter Theori med praktisk Anvendelse, vist sig som en ægte Repræfentant for en af de karakteristiske Ejendommeligheder, der har skaffet britiske Videnskabsmænd et velfortjent særeget Ry.

Maatte ALBION altid
fostre Sönner som
JOHN NAPIER !

I Det Kongelige Danske
Videnskabernes Selskab,
KÖBENHAVN, D: 24" *Juni* 1914.

R. CH. THOMSEN,
Præsident.

H. G. ZEUTHEN,
Sekretær.

Translation

TO THE ROYAL SOCIETY OF EDINBURGH

In honour of the memory of Scotland's celebrated son, the Royal Danish Academy of Sciences sends her congratulation to the Royal Society of Edinburgh on the occasion of the celebration of the tercentenary of the publication of JOHN NAPIER'S *Logarithmorvm Canonis Mirifici Descriptio.*

Through the *magnum opus* of his life, the discovery of logarithms, JOHN NAPIER has not only carried the attempts of earlier generations at facilitating the work of multiplication to a final and complete termination, but he also by determining for the first time a new transcendental function by infinitesimal methods became a predecessor of Newton, the first germs of whose theory of fluxions may be found in NAPIER'S *Constrvctio.* Hence legitimate homage is rendered to NAPIER as one of the pioneers of science.

Through the intimate connexion he established between theory and practical application, he is no less genuine a representative of one of those characteristic features which have deservedly procured a specific fame for British men of science.

May ALBION breed always sons
of the stamp of
JOHN NAPIER!

FROM THE SOCIEDAD CIENTÍFICA 'ANTONIO ALZATE,' MEXICO

We have the honour to inform you that the Scientific Society 'Antonio Alzate,' held a solemn meeting in this city on the 6th instant, to commemorate the Third Centenary of the discovery of Logarithms by the eminent English mathematician John Napier.

The Minister of Public Instruction and Fine Arts presiding, delivered an important address in the presence of the divers delegates representing the various scientific institutions in Mexico.

Engineer J. de Mendizábal y Tamborrel read a paper touching on the life of Napier and his scientific work.

It is a source of great satisfaction to us to bring the abovementioned facts to your knowledge, your Honourable Society having initiated the idea of the Napier Tercentenary Celebration, a work of such transcendency in the progress of science.

We avail ourselves of this occasion to renew to you the assurances of our distinguished consideration.

Mexico, July 26th, 1914.

The President,
ALFONSO PRIMIDA.

The Permanent Secretary,
R. AGUILAR.

THE SECRETARY OF THE
ROYAL SOCIETY OF EDINBURGH,
EDINBURGH.

FROM THE NATIONAL ACADEMY OF SCIENCES, UNITED STATES OF AMERICA

THE NATIONAL ACADEMY OF SCIENCES sends greetings to the ROYAL SOCIETY OF EDINBURGH and gladly joins in its tribute of respect to the memory of JOHN NAPIER.

The inventor of a new instrument of investigation may aid in high degree the development of some branch of science.

But the introduction of logarithms applicable alike in all fields of research, and no less useful in the practical pursuits of life is an achievement of universal appeal.

We congratulate the City of Edinburgh as the birthplace of such a lofty mind, and the academic institutions of Scotland as the worthy conservators and expounders of the potent mathematical methods which have followed this early advance. They may well be proud of that eminent company of men, extending from NAPIER to KELVIN and perpetuated by their able successors of the present day, who have led the van in the attack on the strongholds of nature.

FROM THE UNIVERSITY OF KÖNIGSBERG

DER PROREKTOR DER KÖNIGLICHEN
ALBERTUS-UNIVERSITÄT.

KÖNIGSBERG i. PR., *den* 21 *Juli* 1914.

DER ROYAL SOCIETY OF EDINBURGH danke ich namens der Albertus-Universität für die ehrende Einladung zur 300 jährigen Jubelfeier der Publication von John Napier's *Logarithmorum Canonis Mirifici Descriptio.*

Wenige mathematische Schöpfungen haben die Namen ihrer Urheber so weit getragen wie die Erfindung der Logarithmen. Jeder Schüler lernt mit diesem wichtigsten Rüstzeuge der praktischen Rechnung den Namen Lord Napier's kennen. Dabei ist die genannte Entdeckung nur eine unter vielen unsterblichen Leistungen, die wir diesem Forscher verdanken. Die Feier der Royal Society wird daher in den weitesten Kreisen Wiederhall finden. An jeder Pflanzstätte der Wissenschaft wird das Werk des genialen Schotten auf's neue gerühmt werden.

Die Albertus-Universität zu Königsberg darf einen Grund zu ganz besonderer Anteilnahme in dem Umstande erblicken, dass ein Sohn ihrer Stadt, Peter Crüger (1580-1639), eines der umfassendsten Tafelwerke im Sinne Napier's der deutschen Wissenschaft geschenkt hat.

Möge das Fest einen erhebenden Verlauf nehmen und durch die Untersuchungen, welche es anregen wird, die Wissenschaft befruchten!

Dies sind die Wünsche der Albertina, die sie der hohen Royal Society und den mit dieser zur Feyer zusammentretenden Körperschaften ausspricht.

In ausgezeichnetster Hochachtung ganz ergebenst
der zeitige Prorektor
N. SCHULZE.

FROM THE UNIVERSITY OF MARBURG

AKADEMISCHER SENAT DER KÖNIGL.
UNIVERSITÄT MARBURG.
J.-Nr. 1053

MARBURG, *den* 20 *Juli* 1914.

DER ROYAL SOCIETY OF EDINBURGH

spricht die Königliche Universität Marburg, die lebhaft bedauert, einen Vertreter zu der für den Entdecker der Logarithmen veranstalteten Dreihundertjahrfeier nicht entsenden zu können, auf diesem Wege ihre herzlichsten Glückwünsche aus.

Durch die vor 300 Jahren erfolgte Veröffentlichung des Werkes *Logarithmorum Canonis Mirifici Descriptio* hat uns John Napier das gewaltige Werkzeug der Logarithmen geschenkt, das schon das Staunen der Zeitgenossen erregt und von Jahrhundert zu Jahrhundert von immer wachsender Bedeutung für alle Zweige der Mathematik, für alle Gebiete ihrer Anwendung geworden ist. Mit der Intuition des Genies gab Napier seinem Werk eine solche Form, dass er die Logarithmen zu einer Zeit wissenschaftlich zu begründen vermochte, welche für diese Fundamentierung eigentlich noch nicht völlig reif war, und durch sie wirkte er auf seine grossen unmittelbaren Nachfolger Mercator, Newton, Brook Taylor, Euler, Lagrange so stark ein, dass wir heute das grosse Werk Napier's als eine Säule und Stutze des Gebäudes unserer Analysis, aber auch als das wichtigste Werkzeug in allen Gebieten unseres Denkens und Handelns besitzen, in denen überhaupt die Rechnung als Hülfsmittel gebraucht wird.

TRAEGER.

AN DIE ROYAL SOCIETY OF EDINBURGH,
EDINBURGH.
22 GEORGE STREET.

FROM THE UNIVERSITY OF STRASSBURG

Rektor und Senat der Kaiser Wilhelms-Universität Strassburg an die Royal Society of Edinburgh

Mit grosser Freude haben wir durch Eure gütige Einladung Kenntnis davon erhalten, dass Ihr das Andenken an einen grossen Mann und seine grosse Tat durch ein Fest zu feiern gedenkt, zu dem Ihr alle, denen die Entwickelung der mathematischen Wissenschaften am Herzen liegt, zusammengerufen habt. John Napier's Logarithmorum Canonis Mirifici Descriptio vom Jahre 1614 erfüllte das Suchen einer langen Zeit und veranlasste eine erstaunlich rasch einsetzende und fortschreitende Entwickelung zunächst auf dem Gebiete der Rechenkunst, das, wie andere seiner Entdeckungen beweisen, dem grossen Manne am nächsten lag und hier in der Tat eine vollständige Umwälzung aller Methoden hervorrief. Der von der Erfindung begeisterte unermüdliche Henry Briggs und auf dem Kontinent kein Geringerer als Johannes Kepler liehen dem neuen Gedanken ihre volle Kraft und trugen, was der grübelnde Geist des Lords ersonnen hatte, auf den weiten Plan der Anwendungen. Insbesondere für die Astronomie wurde die Erfindung von unermesslicher Bedeutung, als selbst Keplers Ausdauer kaum mehr imstande war, den Umfang der Rechnungen zu bemeistern. Einen der grössten Wohltäter der Menschheit hat man Lord Napier genannt und niemand wird dieses Urteil freudiger bekennen als der Rechner. Die Astronomie jedenfalls, aber auch viele andere Wissenschaften wären in ihrer Entwickelung ohne das mächtige Hülfsmittel der Napier'schen Logarithmen wesentlich gehemmt worden. Viel langsamer als die praktische wurde die theoretische Bedeutung des Logarithmus gewürdigt; von dem Gebäude, in dem er jetzt ein notwendiger Schlussstein ist, waren nur Steine da und erst der ordnende Riesengeist Eulers setzte den Napier'schen Logarithmus und seine Umkehrung, die Exponentialfunktion, in das Kunstwerk des mathematischen Baues ein.

Der Gedanke, der vor drei Jahrhunderten im Schlosse zu Merchiston keimte, ist Gemeingut aller Kulturvölker geworden; mit Fug und Recht nehmen daher alle an dem bedeutsamen Gedenktag Anteil, der zu einem Tag der Danksagung und Anerkennung aller Mathematiker und Rechner sich gestalten muss.

Wir haben unseren Abgesandten beauftragt, Euch dies zu sagen und senden Euch Gruss und Glückwunsch.

Rektor und Senat
der
Kaiser Wilhelms-Universität
Strassburg
H. Chiari.

FROM THE UNIVERSITY OBSERVATORY, WARSAW

L'Observatoire Astronomique de l'Université de Varsovie a l'honneur d'exprimer ses compliments les plus sincères à la Société Royale d'Edimbourg à l'occasion du trois centième anniversaire de la publication de l'ouvrage éminent de Napier: 'Logarithmorum canonis descriptio, seu arithmeticorum supputationum mirabilis abbreviatio, ejusque usus in utraque trigonometria, ut etiam in omni logistica mathematica amplissimi, facillimi et expeditissimi explicatio, authore ac inventore Joanne Nepero barone Merchistonii, Scoto,' qui a procuré à la science et surtout à l'astronomie un avantage inappréciable, et de souhaiter que la solennité présente, réunissant les savants du monde entier, stimule leur énergie pour le développement ultérieur des idées géniales dans le domaine des sciences physico-mathématiques.

Le Directeur de l'Observatoire,
S. Tscherny.

Varsovie, *le* 20 *Juillet* 1914.

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